3.1 EXPONENTIAL AND LOGISTIC FUNCTIONS

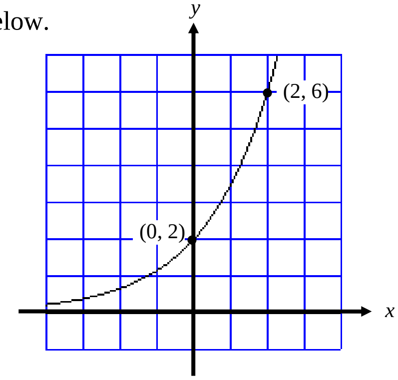
Learning Targets:
1. Recognize exponential growth and decay functions
2. Write an exponential function given the *y*-intercept and another point (from a table or a graph).
3. Be able to define the number *e*
4. Use transformations to graph exponential functions without a calculator.
5. Recognize a logistic growth function and when it is appropriate to use.
6. Use a logistic growth model to answer questions in context.

So far we have discussed polynomial functions (linear, quadratic, cubic, etc.) and rational functions. In all of these functions, the variable *x* was the base. In exponential functions, the variable *x* is the exponent.

Exponential equations will be written as __________________, where *a* = ______________________________.

Exponential functions are classified as growth (if $b > 1$) or decay (if $0 < b < 1$).

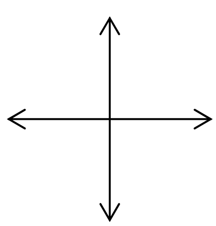
Example 1: Determine a formula for the exponential function whose graph is shown below.



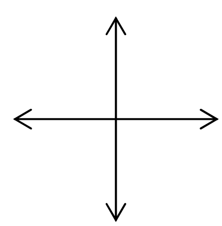
Example 2: Describe how to transform the graph of $f(x) = 3^x$ in to graph of $g(x) = 3^{x+2} - 1$.

Example 3: Sketch the graph of the exponential function.

a) $g(x) = 3^{-x}$



b) $g(x) = -3^x + 2$



The Number e

Many exponential functions in the real world (ones that grow/decay on a continuous basis) are modeled using the base of *e*. Just like $\pi \approx 3.14$, we say $e \approx 2.718$. We can also define *e* using the function $\left(1+\frac{1}{x}\right)^x$ as follows:

$$\text{As } x \to \infty, \ \left(1+\tfrac{1}{x}\right)^x \to e$$

Try convincing yourself that this function approaches *e* using the TABLE function of your calculator.

Do you think it is reasonable for a population to grow exponentially indefinitely?

Logistic Growth Functions … functions that model situations where exponential growth is limited.

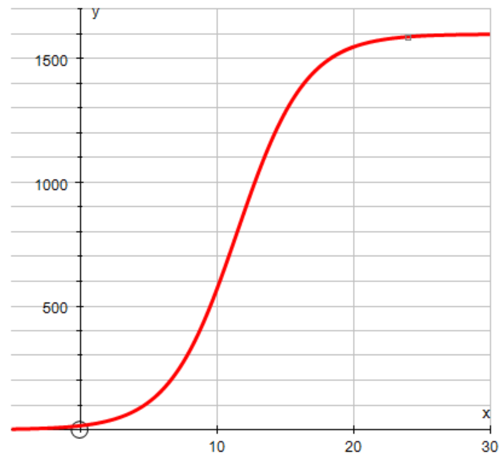
An equation of the form ____________________________ or _______________________________.

where c = ___________________________________.

The graph of a logistic function looks like an exponential function at first, but then "levels off" at $y = c$. Remember from our Parent Functions in chapter 1, that the logistic function has two HA: $y = 0$ and $y = c$.

Example 4: The number of students infected with flu after t days at Springfield High School is modeled by the following function:

$$P(t) = \frac{1600}{1+99e^{-0.4t}}$$



a) What was the initial number of infected students $t = 0$?

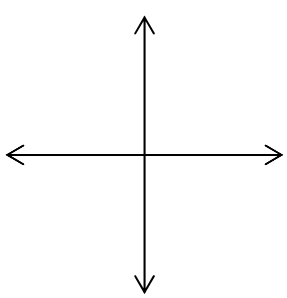
b) After 5 days, how many students will be infected?

c) What is the maximum number of students that will be infected?

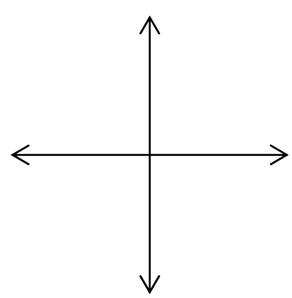
d) According to this model, when will the number of students infected be 800?

Example 5: Find the y–intercept and the horizontal asymptotes. Sketch the graph.

a) $f(x) = \dfrac{9}{1+2(0.6)^x}$



b) $f(x) = \dfrac{8}{1+4e^{-x}}$



3.2 EXPONENTIAL AND LOGISTIC MODELING

Learning Targets:

1. Write an exponential growth or decay model $f(x) = ab^x$ and use it to answer questions in context.
   - Exponential Growth Factor… $b = 1 + \%$ rate (as a decimal)
   - Exponential Decay Factor… $b = 1 - \%$ rate (as a decimal)
2. Write a logistic growth function given the y-intercept, both horizontal asymptotes, and another point.

Example 1: The population of Glenbrook in the year 1910 was 4200. Assume the population increased at the rate of 2.25% per year.

a) Write an exponential model for the population of Glenbrook. Define your variables.

b) Determine the population in 1930 and 1900.

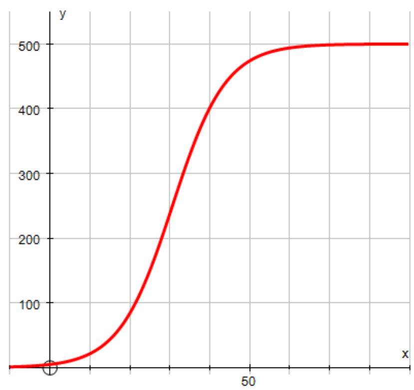
c) Determine when the population is double the original amount.

Example 2: The half-life of a certain radioactive substance is 14 days. There are 10 grams present initially.

a) Express the amount of substance remaining as an exponential function of time. Define your variables.

b) When will there be less than 1 gram remaining?

Example 3: Find a logistic equation of the form $y = \dfrac{c}{1+ae^{-bx}}$ that fits the graph below, if the y-intercept is 5 and the point (24, 135) is on the curve.



3.3 LOGARITHMIC FUNCTIONS AND THEIR GRAPHS

Learning Targets:
1. Rewrite logarithmic expressions as exponential expressions (and vice-versa).
2. Evaluate logarithmic expressions with and without a calculator.
3. Graph and translate a logarithmic function.

Rewriting Logarithmic Expressions

A logarithmic function is simply an inverse of an exponential function. The following definition relates the two functions:

$$\log_b y = x \quad \text{if and only if} \quad b^x = y$$

If you think of the graph of the exponential function $y = b^x$, x could be any real number, but $y > 0$. These same limitations on the variables are true for the logarithm function as well.

Example 1: Evaluate each expression without a calculator.

a) $\log_5 125$ b) $\log_7 1$ c) $\log_{\frac{1}{3}} 81$ d) $\log_8 \sqrt{2}$

Two Special Logarithms

A logarithm with base 10 is called a ____________________ logarithm and is written _________.

A logarithm with base *e* is called a _____________________ logarithm and is written _________.

Example 2: Evaluate each expression without a calculator.

a) $\log \sqrt[4]{10}$ b) $\ln \frac{1}{e}$

A Consequence of the Inverse Properties of Logarithms:

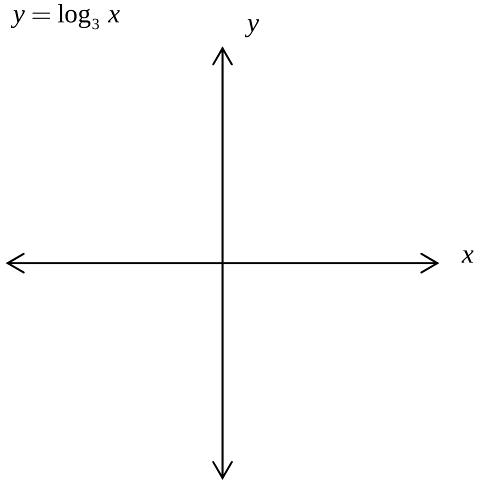
$$b^{\log_b M} = M \quad \dots \text{ and } \dots \quad \log_b\left(b^M\right) = M$$

Example 3: Evaluate each logarithmic expression without a calculator.

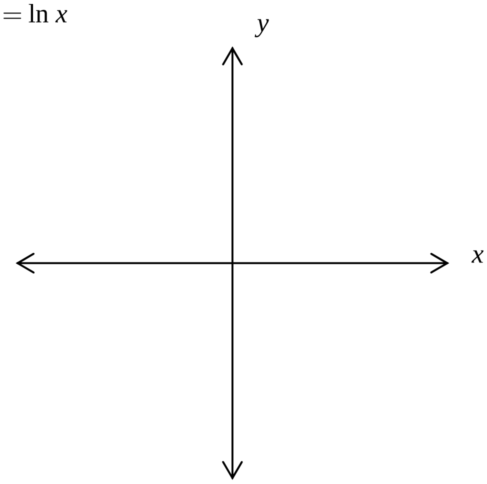
a) $\log_6 6^3$ b) $\log 10^5$ c) $e^{\ln 5}$ d) $8^{\log_8 7}$

Example 4: Graph the following functions without a calculator.

a) $y = \log_3 x$

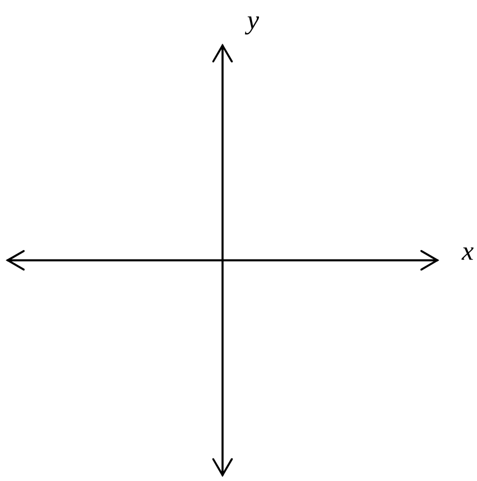


b) $y = \ln x$

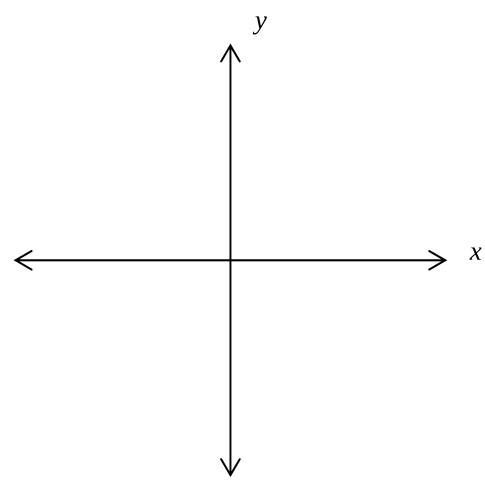


Example 5: Graph the following transformations of the two functions above without a calculator.

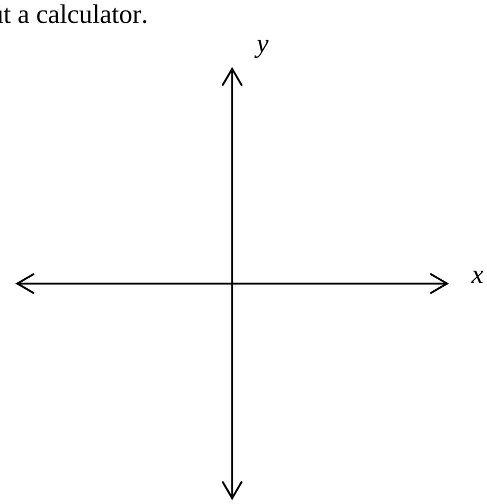
a) $y = \log_3 (x-3)+1$



b) $y = \ln(-x)$



Example 6: Graph $y = \log_3 (2-x)$ without a calculator.



3.4 PROPERTIES OF LOGARITHMS

Learning Targets:
1. Use properties of logarithms to expand a logarithmic expression.
2. Use properties of logarithms to write a logarithmic expression as a single logarithm.
3. Evaluate logarithms with bases other than *e* or 10 using a calculator.

Properties of Logarithms … these properties follow from the properties of exponents

3 Rules of Logarithms:
1. $\log_b(MN) = \log_b(M) + \log_b(N)$
2. $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$
3. $\log_b\left(M^k\right) = k \cdot \log_b(M)$

Example 1: Expand each logarithmic expression.

a) $\log\left(xy^3\right)$

b) $\ln\left(\frac{x^2\sqrt[3]{q}}{y\sqrt{t}}\right)$

Example 2: Condense each logarithmic expression into a single logarithm.

a) $3\ln x - \frac{1}{2}\ln y + 5\ln z$ b) $3\ln x - \left(\frac{1}{2}\ln y + 5\ln z\right)$ c) $\frac{4}{3}\log_b 27 - 2\log_b 9$

Your calculator (unless you have the new TI-84+ operating system) will only evaluate a logarithm with base 10 or *e*. If you need to evaluate a logarithm with a different base, you must use the change of base formula.

Change of Base Formula: $\log_b a = \frac{\log a}{\log b}$ or $\frac{\ln a}{\ln b}$

Example 3: Evaluate each logarithmic expression using your calculator.

a) $\log_7 5$

b) $\log_{0.4} 8$

3.5 EQUATION SOLVING AND MODELING

Learning Targets:
1. Solve exponential and logarithmic equations.

When you solve an equation, you "undo" what has been done … addition to undo subtraction, multiplication to undo division. Since exponents and logarithms are inverses of each other, it follows that in order to solve a logarithmic equation, you can write it as an exponent to "undo" the logarithm, and if you are solving for an exponent, you write the equation as a logarithm.

NOTE: You can only switch between exponential and logarithmic forms when you have $\log_b y = x$ or $b^x = y$

Example 1: Solve the following equations:

a) $e^{x-2} = 19$

b) $3-5(2)^{3x} = -14$

c) $\log_2(x) = 4$

d) $7-3\log(x) = -5$

e) $\log x + \log(x+21) = 2$

f) $\log_3(x+4) - \log_3(x-5) = 2$

3.6 MATHEMATICS OF FINANCE

Learning Targets:
1. Apply correct compounding interest formulas (either continuously or n times per year)
2. Use the FINANCE application on your TI-83 or 84 calculator to solve for unknown quantities.

Compounding Interest "n" Times Per Year

When the interest earned is added to the original amount and then interest is calculated on that interest (in other words, you are earning interest on your interest), we say the interest is compounded. The formula for compounding interest n times each year is given by

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Where … A = new amount
P = original amount or Principle
r = the interest rate as a decimal
n = the number of times a year interest is calculated
t = the number of years

Example 1: Consider the situation where $n = 1$.

a) Explain what this means in the context of compounding interest.

b) What does the compounding interest rate look like when $n = 1$?

Example 2: Victor deposits $500 into a savings account earning 6% interest compounded *monthly*. How much money is in his account in 12 years?

Compounding Interest Continuously

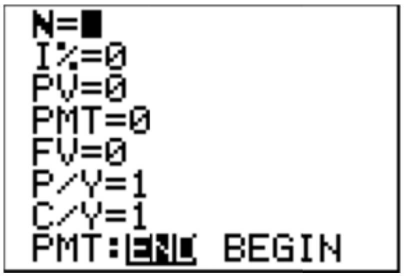
The formula for compounding interest continuously is ________________________. Notice in this formula, you need the interest rate r and NOT the growth factor b.

Example 5: Bob invests $500 into a savings account earning 6% interest compounded *continuously*. How much money does Bob have in 12 years?

Using the FINANCE App On Your TI-83 or TI-84 Calculator

Your graphing calculator has a very powerful financial calculator application built into its programming called the TMV Solver. TMV stands for *Time–Value–of–Money*. The application solves for one variable when given values for any four of the five remaining variables. It is especially useful when you are making the same periodic payment (or investment) more than once a year…i.e. car payment.

First we need to understand the 5 variables involved…Press APPS, Finance, TMV Solver to get started…



1. n = the number of compounding periods
2. I% = the annual percentage rate
3. PV = the Present Value of the account or investment
4. PMT = the payment amount
5. FV = the Future Value of the account or investment
6. P/Y = payments per year
7. C/Y = compounding periods per year…for our use C/Y = P/Y
8. PMT: END BEGIN means payment at the end or beginning of the payment period.

NOTE: Enter any money received in as positive numbers and any money paid out as negative numbers.

Example 6: Sal opened a retirement account and pays $50 at the end of each quarter (4 times/year) into it. The account earns 6.12% annual interest. Sal plans to retire in 15 years. How much money will be in his account at that time?

Example 7: Kim wants to buy a used car for $9000. The bank offers her a 4-year loan with an APR of 7.95%. What will Kim's monthly car payment be? Assume she makes her payment on the 1st of each month.