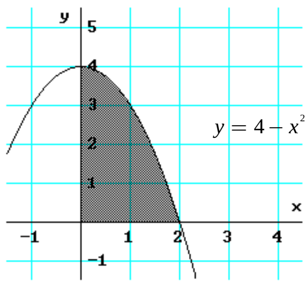
4.4 Properties of Definite Integrals

Previously, we found the approximate area under the curve by creating rectangles and adding the areas of all these rectangles. In the last section we determined that the definite integral gives us the area under the curve as long as the curve is above the *x*-axis. The following will hopefully show you how these two ideas relate and maybe to get a brief look into the theory of why the definite integral is viewed as the limit of a Riemann Sum.

Example: Consider the function $y = 4 - x^2$. Find the area of the region found in the first quadrant below.



a) Instead of drawing multiple rectangles, draw ONE rectangle that will represent all the others.

b) The height of this rectangle will be ___________.

c) The width of this rectangle will be ___________.

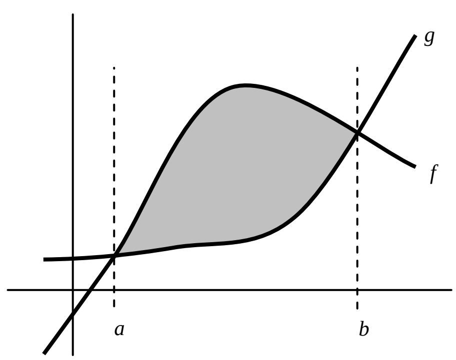
d) Thus, the area of this single rectangular strip is _______________.

e) Since a definite integral is simply the sum of an infinite number of these infinitely thin rectangles, the total area of the shaded region above can be written as

f) Calculate the area of the shaded region.

We can apply this same concept to the *area between curves*. Consider the two functions *f* and *g* below.

First: Draw a rectangular strip. What is the height and width of your rectangle? Would the height and width of the rectangle strip be different if you drew it in a different place?



Second: The area between the curves is approximately the sum of an infinite number of these infinitely thin rectangles. Therefore, the area between these curves can be written as

Example: Find the area of the region bounded by the graphs of $y = x^2 + 2$, $y = -x$, $x = 0$, and $x = 1$.

Step 1: Draw a picture and shade the desired region.

Step 2: Draw an arbitrary rectangular strip.

Step 3: Using the area of the rectangular strip as a guide, set up and solve an integral to find the area between the curves.

Properties for Definite Integrals

1. Order of Integration: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

If you reverse the *order* of integration you get the opposite answer.

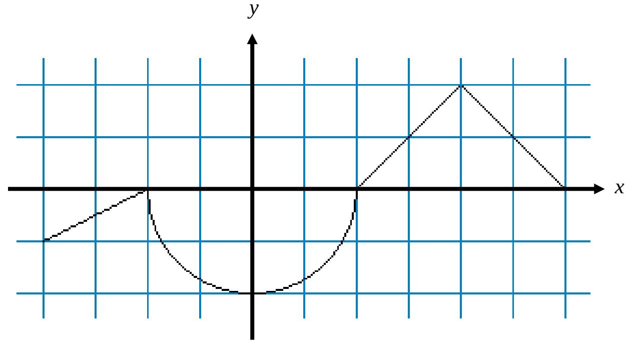
2. Zero: $\int_a^a f(x)\,dx = 0$

This should make sense if you think about the "area" of a rectangle with no width.

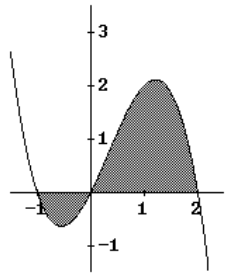
3. Additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Pay close attention to the limits of integration … this comes in handy when dealing with total area or other functions where we need to break them into smaller parts.

Example: In the last section, we used $\int_{-4}^{6} |f(x)|\,dx$ to find the total area between the x – axis and the graph between $x = -4$ and $x = 6$. Without using absolute value signs, write an expression that can be used to find the total area between the x – axis and the function between $x = -4$ and $x = 6$.



Example: Find the area of the shaded region of the graph below if $f(x) = 2x + x^2 - x^3$.



Average Value of a Function

Example: If you were asked to average 28 test scores, how would you do it?

Example: Suppose *f* (*t*) represents the temperature outside of PPCC at time *t*, measured in hours since midnight. If I asked you to find the average temperature for the next 24 hours, how would you do it?

One way to start is to measure the temperature at *n* equally spaced times $t_1, t_2, t_3, \ldots t_n$ and then average those temperatures.

Example: Using this method, write an expression for the AVERAGE temperature.

The larger the number of measurements, the more accurately this will reflect the average temperature. Notice we can write this expression as a Riemann sum by first noting that the interval between measurements will be $\Delta t = \frac{24}{n}$, so $n = \frac{24}{\Delta t}$.

Example: Substitute this value of *n* into your expression above and simplify.

Example: The last expression gives us an approximate Average Temperature. As $n \to \infty$ (meaning we are taking a lot of temperature readings) this Riemann Sum becomes a definite integral. Write the Definite Integral that gives us the Average Temperature since midnight.

The process that we just used to find the average temperature is used to find the Average Value of any function.

The Average Value of a Function

If f is integrable on $[a, b]$, its **average value** on $[a, b]$ is given by

$$\text{AVERAGE VALUE} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{\int_a^b f(x)\,dx}{b-a}$$

Just remember … **"Average Value = Integral Over Interval."**

Example: A keyboarder's speed over a 5 minute interval is given by $W(t) = -6t^2 + 12t + 90$, where $W(t)$ is the speed in words per minute and $0 \leq t \leq 5$.

a) Find the speed at the beginning of the interval.

b) Find the maximum speed and when it occurs.

c) Find the total number of words typed in the 5 minute interval.

d) Find the average speed over the 5 minute interval.