2.7 IMPLICIT DIFFERENTIATION AND RELATED RATES

Implicit and Explicit Functions

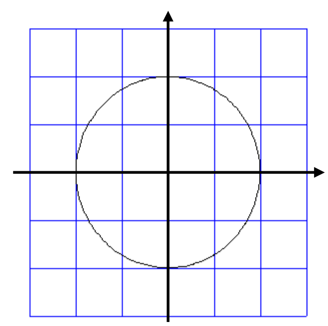
Suppose your boss says, "I have had it with your incompetence. You've screwed up everything we've ever given you to do! The entire company is on the brink of bankruptcy, single-handedly thanks to you! Now, clean out your desk and don't show your face here again!"
You might argue that the boss never actually said, "You're fired." There was not explicit dismissal ever stated, but you have to face the facts: It was certainly implied!

Equations can be written with implied meanings as well. For instance, the equation $x + 2y - 3 = 0$, implies that y is a function of x, even though it is not written in the form $y = -\frac{1}{2}x + \frac{3}{2}$. Up to this point in this class we have been using functions of x expressed in the form $y = f(x)$ such as

$$y = \frac{x+1}{x+2} \qquad \text{or} \qquad y = \sin x\,.$$

An equation of this form is said to define y as an *explicit* function of x.
If we have an equation that involves both x and y in which y has not been solved for x, then we say the equation defines y as an *implicit* function of x. In this case, we may (or may not) be able to solve for y in terms of x to obtain an explicit function (or possibly several functions).

Example: Solve the equation $x^2 + y^2 = 4$ to obtain y written as an explicit function of x. The graph of this equation is a circle. What is the graph of each explicit function?



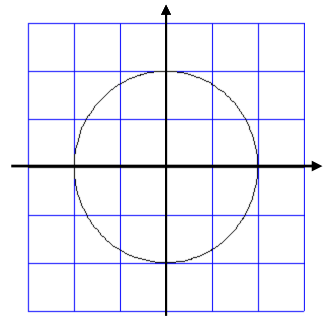
If we have y written as an explicit function of x, $y = f(x)$, then we know how to compute the derivative $\frac{dy}{dx}$. For an equation which defines y as an implicit function of x, we can compute the derivative $\frac{dy}{dx}$ without solving for y in terms of x with the following procedure. **The key to this entire procedure is to remember that even though you did not (or cannot) write *y* as a function of *x*, *y* is implicitly defined as a function of *x*.**

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation **with respect to *x*.** Remember, y is a function of x (use the Chain Rule)
2. Collect all $\frac{dy}{dx}$ terms on the left side of the equation and move all other terms to the right side of the equation.
3. Factor $\frac{dy}{dx}$ out of the left side of the equation.
4. Solve for $\frac{dy}{dx}$. (It is okay to have both x's and y's in your answer)

To find $\frac{dy}{dx}$ at a given point, plug both the x and y value into the equation you obtained in step 4.

Example: Find $\frac{dy}{dx}$ for the circle $x^2 + y^2 = 4$:

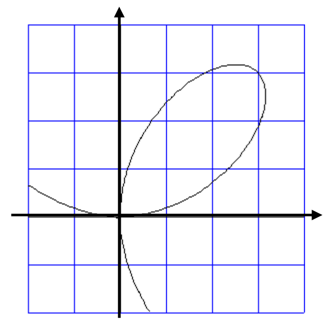


a) by differentiating the explicit functions of x

b) by differentiating implicitly

c) Find where the derivative is positive, where it is negative, where it is zero, and where it is undefined.

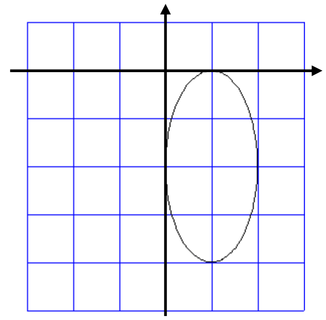
Example: Given the curve $x^3 + y^3 = 6xy$ (shown to the right).



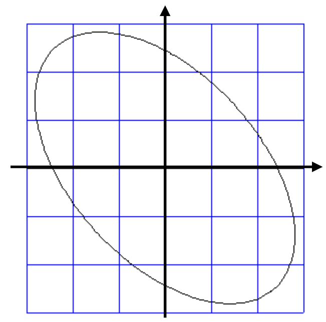
a) Find $\frac{dy}{dx}$.

b) Find the equation of the tangent line to the graph at the point $\left(\frac{4}{3}, \frac{8}{3}\right)$.

Example: Find the points at which the graph of $4x^2 + y^2 - 8x + 4y + 4 = 0$ has a vertical tangent line. (Pretend the picture isn't there until *after* you have found the points!)



Example: Find the point(s) (if any) of horizontal tangent lines: $x^2 + xy + y^2 = 6$



Related Rates

Example: Suppose both *y* and *x* are differentiable functions of *t*. Differentiate $y = x^2$ with respect to *t*. (Find $\frac{dy}{dt}$).

a) by using differentials

b) by implicitly differentiating *x* and *y* with respect to *t*.

Example: If you wanted to solve for $\frac{dy}{dt}$, what other information would have to be given to you?

Example: Suppose you are told that the particle moving along the curve $y = x^2$ is moving horizontally at 2 cm/s. Find the rate of change in the particle's vertical position at the exact moment the particle is at (3, 9).

In a related rates problem, you have an equation relating two or more things that *change over time*, and we want to find the rate of change (a derivative) of one of these things. It is important to understand that without some conditions given to use, we cannot solve the problem.

Guidelines for Solving Related-Rate Problems

Step 1: Read the problem, really! You'd be amazed how many people skip this step. Then read it again! ☺

Step 2: Draw a diagram showing what's going on. Identify all relevant information and assign variables to what's changing. Use the general case (numbers for values that NEVER change in this situation, and variables for anything that is changing).

♬ : Related Rates usually involve motion … any diagram you draw is like a still picture of what is occurring. Any part of your picture that NEVER changes can be labeled with a constant (or number), but any part of your picture that is in motion or is changing MUST be labeled with a variable!

In other words, if the radius of a circle is increasing and you are asked to find the rate of change in the area at the exact moment when the radius is 5 cm, then your diagram would be a circle, but you would NOT label the radius 5 because it is changing … you would label the radius r.

Step 3: Find the equation that gives the relationship between the variables you just named in step 2.

Step 4: Find the particular information (values of variables at the exact moment you drew your diagram) for the problem and write it down. Then, list what you are looking for (normally this would be a derivative).

Step 5: Implicitly differentiate the equation with respect to time t. Usually this equation will have at least two derivatives. If it has more than two, be sure you have enough information, *or* you may have to find a relationship between two of the variables, and rewrite the equation in step 3 using this relationship.

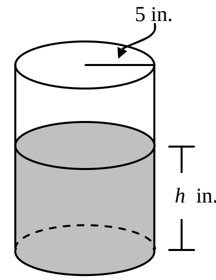
Step 6: Plug in the particular information, and solve for the desired quantity. DO NOT DO THIS UNTIL AFTER YOU HAVE TAKEN THE DERIVATIVE!

Step 7: Write down your answer and circle it with your favorite color. (be sure to use correct units)

Example: Tweety is resting in a bird house 24 feet off the ground. Using a 26 foot ladder which he leaned against the pole holding the bird house, Sylvester tries to steal the small yellow bird. Tweety's bodyguard, Hector the dog, starts pulling the base of the ladder away from the pole at a rate of 2 ft/s. How fast is the ladder falling when it is 10 feet off the ground?

Example: A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure to the right. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.



Example: A baseball diamond has the shape of a square with sides 90 feet long. Tweety is just flying around the bases, running from 2nd base (top of the diamond) to third base (left side of diamond) at a speed of 28 feet per second. When Tweety is 30 feet from third base, at what rate is Tweety's distance from home plate (bottom of diamond) changing?