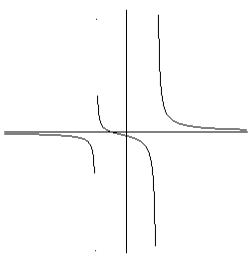
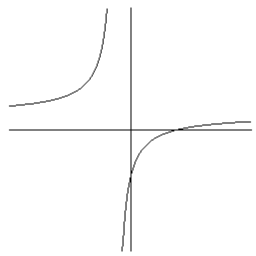
2.3 GRAPH SKETCHING: ASYMPTOTES AND RATIONAL FUNCTIONS

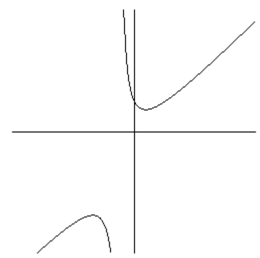
All the functions from the previous section were continuous. In this section we will concern ourselves with rational functions which are not continuous.

Definition: A Rational Function is a function that looks like $\frac{p(x)}{q(x)}$, where *p* and *q* are polynomial functions.

The graphs below illustrate examples of rational functions generated by a computer.







As you can see from the pictures, rational functions have vertical, horizontal, and slant asymptotes.

Example: The first graph above has ____ vertical asymptotes and ____ horizontal asymptotes. Draw them.

Example: The second graph above has ____ vertical asymptotes and ____ horizontal asymptotes. Draw them.

Example: The third graph above has ____ vertical asymptotes and ____ slant asymptotes. Draw them.

Our GOAL: Find the asymptotes of a rational function and draw a sketch without using the graphing part of our calculator.

Vertical Asymptotes:

Pre-Calculus definition: The vertical line $x = c$ is a vertical asymptote if you plug $x = c$ into the equation and you get a nonzero value divided by zero.

Calculus definition: The vertical line $x = c$ is a vertical asymptote if the function approaches positive or negative infinity as *x* approaches *c* from the left or from the right.

Using limit notation … we have the line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

Horizontal Asympototes:

The horizontal line $y = b$ is a horizontal asymptote if the function approaches *b* as *x* approaches positive or negative infinity.

Using limit notation … we have the line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

Example: Find the vertical asymptotes of each function.

a) $f(x)=\dfrac{x^2-1}{2x+4}$

b) $f(x)=\dfrac{1-x}{2x^2-5x-3}$

c) $f(x)=\dfrac{x-2}{3x^2-5x-2}$

Horizontal Asymptotes and End Behavior

The *end behavior* of a function is how the function "behaves" as *x* approaches infinity (a.k.a. "the end")

The *end behavior* of a polynomial is simply the leading coefficient and the variable with the highest degree. For instance, the end behavior of $5-3x+8x^2-9x^3$ is $-9x^3$.

Since a rational function is just two polynomial functions divided, we can look at the end behavior of a rational function simply as the end behavior of the polynomial in the numerator divided by the end behavior of the polynomial in the denominator.

Example: $\lim\limits_{x\to\infty}\dfrac{1}{x}=$

Example: $\lim\limits_{x\to\infty}\dfrac{2x+5}{3x^2-6x+1}=$

Example: $\lim\limits_{x\to\infty}\dfrac{2x^2-3x+5}{x^2+1}=$

Example: $\lim\limits_{x\to\infty}\dfrac{x^4+x^3+9}{3x-3}=$

The last example did not have a limit as *x* approaches infinity. Thus we look for slant asymptotes.

If the numerator has <u>higher degree</u> than the denominator, use long division to find the slant asymptotes.

Example: Find the slant asymptote of the function $f(x) = \dfrac{x^2 - 4}{x + 7}$

Example: Consider the graph $g(x) = \dfrac{x+1}{x^2 - 2x - 3}$.

a) Find any horizontal asymptote(s).

b) Find any vertical asymptote(s).

Example: Sketch the function that satisfies the stated conditions.

$\lim_{x\to 1} f(x) = 2$

$\lim_{x\to 5^-} f(x) = \infty$

$\lim_{x\to 5^+} f(x) = \infty$

$\lim_{x\to \infty} f(x) = -1$

$\lim_{x\to -\infty} f(x) = 0$

$\lim_{x\to -2^-} f(x) = \infty$

$\lim_{x\to -2^+} f(x) = -\infty$

Putting it all together

Example: Sketch the following function: $h(x)=\dfrac{x-2}{x+1}$

a) Find any horizontal asymptote(s).

b) Find any vertical asymptote(s).

c) Find where *h* (*x*) is increasing and decreasing, and find any relative maximums and minimums.

d) Find where *h* (*x*) is concave up or concave down, and find any inflection points.

e) Find the *x* and *y* intercepts (if they exist).

f) GRAPH USING ALL THIS INFORMATION. Find more points to make your sketch more accurate.

Example: Sketch $y=\frac{x^2-7x+6}{x^2+x-2}$

a) Find the horizontal asymptote.

b) Find the values of *x* that make the denominator 0. Plug these values into the numerator.

c) What is the vertical asymptote? What happens on the graph at the other value of *x* you found in part *b*?

d) Factor the numerator and denominator. Find a "simpler" version of this function.
The "simpler" version is the same as the original except at 1 point

e) Using the "simpler" function, find where the function is increasing, decreasing, and where any relative maximums and minimums occur.

f) Using the "simpler" function, find where the function is concave up, concave down, where any points of inflection occur.

g) Find the *x* and *y* intercepts (if they exist).

h) GRAPH this function using this information.

Example: Sketch the graph of $q(x) = \dfrac{2x^2}{x-4}$.

a) Find the asymptotes. (horizontal, vertical, or slant?)

b) Find where $q(x)$ is increasing, decreasing, and where any relative maximums and minimums occur.

c) Find where $q(x)$ is concave up, concave down, and where any points of inflection occur.

d) Find the x and y intercepts (if they exist).

e) GRAPH the function.