1.7 THE CHAIN RULE

Suppose you were asked to differentiate

$$h(x)=\sqrt{x^2+1}\,.$$

Up to this point in the course, we have no tools with which to differentiate this function.

This function is an example of a function which can be found by the composition of two functions.
The notation for *f* composed with *g* is given by

$$(f\circ g)(x)=f\left(g(x)\right)$$

This is read "*f* circle *g* of *x* equals *f* of *g* of *x*".
In other words, a composite function has one function INSIDE another function. [*g* is INSIDE *f* in the notation above]

Example: Let $f(x)=x^2+1$ and $g(x)=\sqrt{x}$.

a) Find $g\left(f(x)\right)$

b) Find $f\left(g(x)\right)$

The Chain Rule

The derivative of a composite function $f\left(g(x)\right)$ is given by

$$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)g'(x)$$

Composite functions have an "inside function" and an "outside function". Another way to look at this would be

$$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)g'(x)$$

Derivative of the "outside function" … leave the "inside function" alone.

Derivative of the "inside function"

The toughest part (at first) is learning to identify the "inside" and "outside" functions.

Example: Using *h* (*x*), from the top of the page, identify the "inside" and "outside" functions, then find $h'(x)$.

Another way to apply the chain rule is to break the inside and outside parts into their separate pieces. For example, with the function we used before, notice we could let

$$y = \sqrt{u} \text{ , and}$$
$$u = x^2 + 1.$$

Differentiating each of the above functions separately, we obtain

$$\frac{dy}{du} =$$
$$\frac{du}{dx} =$$

The Chain Rule then allows us to find $\frac{dy}{dx}$ simply by multiplying $\frac{dy}{du} \cdot \frac{du}{dx}$. Thus, $\frac{dy}{dx} =$

Example: Each of the following examples can be done without using the chain rule. First state how to find the derivative without using the chain rule, and then use the chain rule to differentiate.

(a) $f(x) = \frac{2}{3x+1}$

(b) $g(x) = (x+2)^3$

Example: Find $\frac{dg}{dt}$, if $g = \left(\frac{t-2}{2t+1}\right)^9$

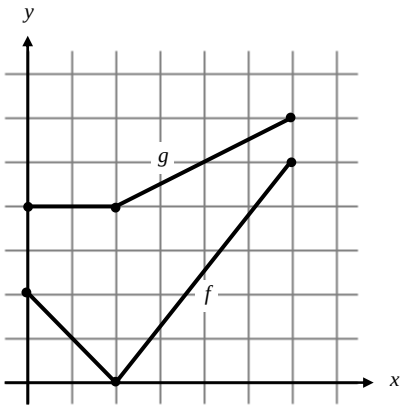
Example: Find $k'(x)$, if $k(x) = \sqrt{2x^3 - 3}$.

Example: Differentiate the function $y = 12(2x+1)^{2/3}(3x-4)^{5/4}$

Example: Let $r(x) = f(g(x))$ and $s(x) = g(f(x))$ where *f* and *g* are shown in the figure at right.

a) Find $r'(1)$.

b) Find $s'(4)$.



Example: Find the derivative of each function below.

a) $y = -9x + 3$

b) $y = x^2 - 7x + 3$

c) $y = x^{1/4}$

d) $y = \dfrac{3}{x^6}$

e) $f(x) = \sqrt[3]{2x^5 - 8}$

f) $f(x) = \dfrac{5x^3 + 4}{(2x-1)^3}$

g) $f(x) = (x-3)\sqrt{x^2 + 2}$