1.6 DIFFERENTIATION TECHNIQUES: PRODUCT AND QUOTIENT RULES

We know that the derivative of the sum of two functions is the sum of the derivatives of the two functions. This does not work for the product and quotient of two functions. To illustrate this, we look at the following example.

Example: Find $\frac{d}{dx}\left[x^2 \cdot 3x\right]$.

a) First … multiply the functions inside the bracket, then take the derivative.

b) Second … take the derivative of each function and multiply them together … do you get the same result?

<u>*Rule 5*</u>: *The Product Rule*

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]$$

This can also be written as

$$\frac{d}{dx}[uv] = uv' + vu'$$

If we view *u* and *v* as differentiable functions of *x*.

For polynomial functions it is not always necessary to use the product rule, however, with square root, exponential, logarithmic, and other functions, it is often a necessary tool.

Example: Let $f(x) = \left(3x^3 + 4x^2\right)\left(2x^4 - 5x\right)$. Find $f'(x)$ without using the product rule first, then using the product rule.

Example: Let $y = \left(3 + 2\sqrt{x}\right)\left(5x^3 - 7\right)$. Find $\frac{dy}{dx}$.

Example: Find the equation of the tangent line to the graph of $f(x) = (x^3 - 3x + 1)(x + 2)$ at the point (1, –3).

Since we've been multiplying … let's try dividing. The rule for taking the derivative of two functions that are divided looks intimidating at first, but it's really not that bad … once you get to know it.

Rule 6: *The Quotient Rule*

As long as $g(x) \neq 0$,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Again, if we consider *u* and *v* to be differentiable functions of *x*, then this is also written as

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}.$$

Looks GREAT! Doesn't it?! Well, luckily for you, it ain't that bad. With thanks to Snow White and the Seven Dwarfs, if we replace *u* with *hi* and *v* with *ho* (*hi* for high up there in the numerator and *ho* for low down there in the denominator), and letting *D* stand for "the derivative of", the formula becomes

$$D\left(\frac{hi}{ho}\right) = \frac{ho\,D(hi) - hi\;D(ho)}{(ho)^2}$$

In words, that is "**ho dee hi minus hi dee ho over ho ho**". Now, if Sleepy and Sneezy can remember that, it shouldn't be any problem for you.

Example: Find $\frac{d}{dx}\left(\frac{x}{x^2+1}\right)$

Example: Find $\frac{d}{dx}\left[\frac{5x^2}{x^3+1}\right]$