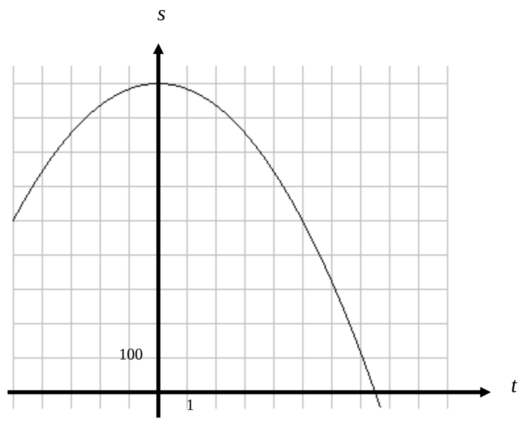
1.4 DIFFERENTIATION USING LIMITS OF DIFFERENCE QUOTIENTS

At the risk of being redundant … let's go back to Wile E. Coyote jumping off the cliff.

$$s(t) = -16t^2 + 900$$

A graph of this equation is shown below.



In the last section, we were able to find Wile E.'s average velocity from $t = 0$ to $t = 5$ seconds.

If we wanted to find the velocity of Wile E. Coyote at *exactly* 5 seconds, we tried to determine the average velocity using values of t that were closer and closer to $t = 5$.

Example: Find Wile E.'s average velocity (rate of change) from $t = 4$ to $t = 5$ seconds. Graphically show this above.

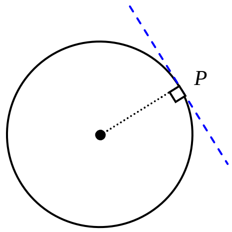
Example: Find Wile E.'s average velocity (rate of change) from $t = 4.5$ to $t = 5$ seconds. Graphically show this above.

Example: Find Wile E.'s average velocity (rate of change) from $t = 4.9$ to $t = 5$ seconds. Graphically show this above.

Example: What do you think would be the graphical interpretation of the velocity at exactly 5 seconds?

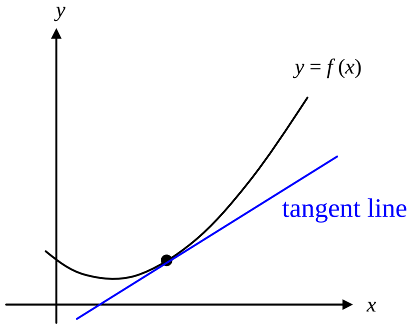
Tangent Lines

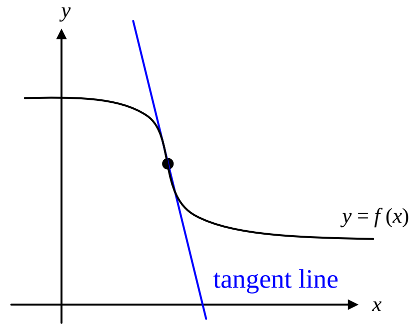
For a circle, the tangent line at a point P is the line that "touches the circle" only once.

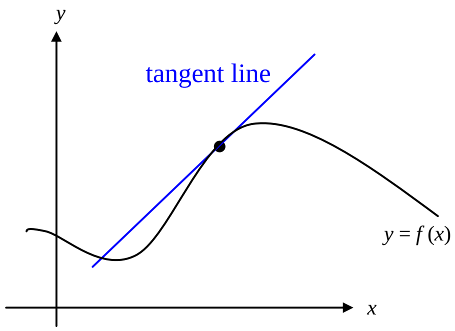


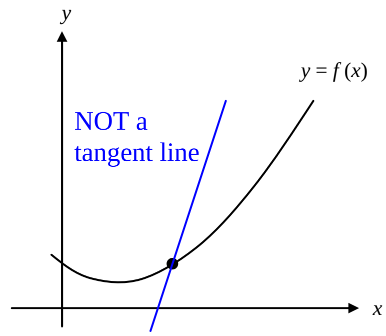
For a general curve, however, the problem of defining a tangent line is more difficult.

If we define a tangent line to the curve in the same way we define a tangent to the circle, sometimes it works, sometimes it doesn't.









Definitions that work for some cases but not for others are not acceptable to mathematicians. So we need another way to define a tangent line.

Earlier we determined that to write the equation of a line we need 2 things. A ___________ and a ________________.

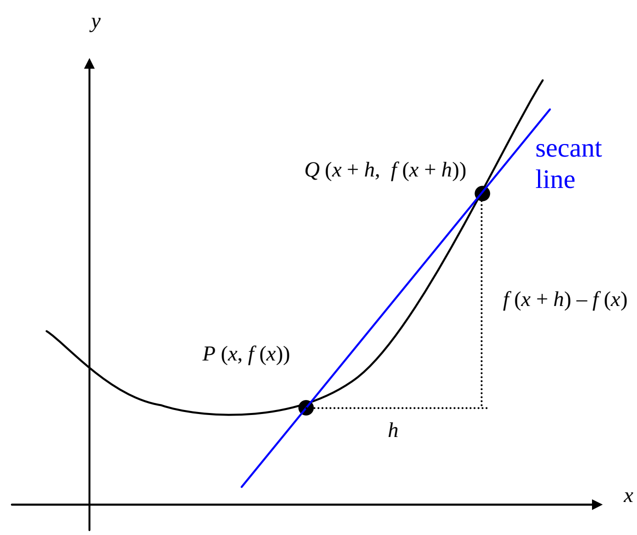
Since we will be finding the tangent line to a curve *at a given point*, the first part is already done.

To find the equation of a tangent line we only need to figure out a way to calculate the slope.

We can *approximate* the slope of the tangent line using a **secant line**, and our difference quotient from before.

If $P\ (a, f(a))$ is the point of tangency we are concerned with, then we can pick an arbitrary point Q on the graph and estimate the tangent line at P using the slope of the secant line through P and Q.

Example: What is the slope of the secant line?



The beauty of this graph is that you can obtain more and more accurate approximations to the slope of the tangent line by choosing points closer and closer to the point of tangency.

How do we get closer and closer to the point of tangency?

Draw at least 3 more secant lines, using a point closer to P each time.

As $h \to 0$, the slope of the secant line approaches the slope of the tangent line.

The slope of a line is always constant. The slope of a curve is constantly changing. Think of a curve as a roller coaster that you are riding. If for some reason the "track" were to just disappear, you would go flying off in the direction that you were traveling at that last instant before the track disappeared. The direction that you flew off to would be the slope of the curve at that point.

Using the slopes of the secant lines from the last example leads us to the following definition.

Slope of a Tangent Line

The slope of a tangent line at the point $(x, f(x))$ is given by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This limit is also the **instantaneous rate of change** of $f(x)$ at x.

If average rate of change is the slope of a secant line, then instantaneous rate of change is the slope of the tangent line.

Example: Find the instantaneous rate of change of $s(t) = -16t^2 + 900$ when $t = 5$.

a) Write the difference quotient.

b) Simplify the difference quotient.

c) Find the limit as h approaches 0.

d) Plug in $t = 5$.

The Derivative

For a function $y = f(x)$, its **derivative** at x is the function $f'(x)$ … read "f prime of x" … defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that the limit exists.

If $f'(x)$ exists, then we say that f is **differentiable** at x.

Steps to finding the derivative:

1. Write the difference quotient.
2. Simplify the difference quotient.
3. Find the limit as h approaches 0.

Example: Consider the function $f(x) = 3x + 2$. Find $f'(x)$

a) Write the difference quotient.

b) Simplify the difference quotient.

c) Find the limit as h approaches 0.

d) Why would you NOT want to use this method for this particular type of function in the future?

Example: Consider the function $k(x) = \frac{7}{x}$. What is $k'(3)$?

When Do Derivatives Fail To Exist?

Since a derivative is defined as a limit, the derivative fails when the limit fails. So, let's refresh our memories on why limits fail.

#1: Limits fail when the function approaches infinity as x approaches a.

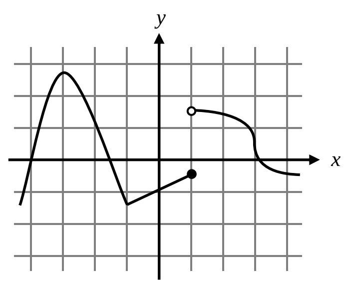
When considering slopes, this occurs when the tangent line is _______________________.

#2: Limits fail when the function approaches different values from the right and left.

When considering slopes, this occurs when there is a ________________________.

Derivatives also fail to exist when a function is not continuous.

Example: List the points in the graph at which the function below is not differentiable.



Example: Are there any points where the function is continuous but not differentiable?

Example: Are there any points where the function is differentiable but not continuous?

Example: Find an equation of the tangent line to the graph of $p(x) = 4 - x^2$ when $x = 3$.