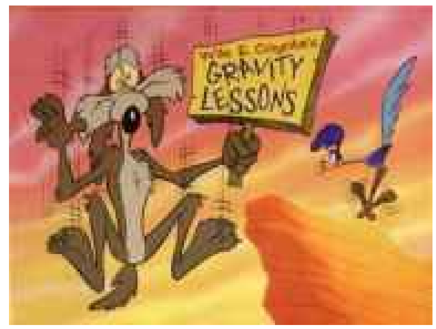
1.1 LIMITS: A NUMERICAL AND GRAPHICAL APPROACH

Limits are what separate Calculus from pre – calculus. Using a limit is also the foundational principle behind the two most important concepts in calculus, derivatives and integrals. Limits can be found using substitution, graphical investigation, numerical approximation, algebra, or some combination of these.

Average and Instantaneous Velocity

In pre – calculus courses, you used the formula $d = rt$ to determine the speed of an object. What you found was the object's average speed. A moving body's **average speed** during an interval of time is found by dividing the total distance covered by the elapsed time. (Speed is always positive … Velocity indicates direction and can be negative.) We are going to find the **average velocity**.

If an object is dropped from an initial height of h_0, we can use the position function $s(t) = -16t^2 + h_0$ to model the height, *s*, (in feet) of an object that has fallen for *t* seconds.



Example: Wile E. Coyote, once again trying to catch the Road Runner, waits for the nastily speedy bird atop a 900 foot cliff. With his Acme Rocket Pac strapped to his back, Wile E. is poised to leap from the cliff, fire up his rocket pack, and finally partake of a juicy road runner roast. Seconds later, the Road Runner zips by and Wile E. leaps from the cliff. Alas, as always, the rocket malfunctions and fails to fire, sending poor Wile E. plummeting to the road below disappearing into a cloud of dust.

a) What is the position function for Wile E. Coyote?

b) Find Wile E.'s average velocity for the first 3 seconds.

c) Find Wile E.'s average velocity between $t = 2$ and $t = 3$ seconds.

d) Find Wile E.'s velocity at the instant $t = 3$ seconds.

The problem with part *d* is that we are trying to find the *instantaneous velocity*. Without the concept of a limit, we could not find the answer to part *d*. Using a limit to solve this problem involves studying what happens as we get "close" to 3 seconds.

Example: Find the average velocity between $t = 2.5$ and $t = 3$ seconds.

Example: Find the average velocity between $t = 2.9$ and $t = 3$ seconds.

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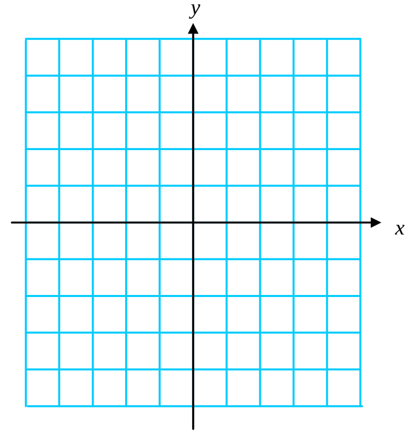
So, even though we cannot find the average velocity at exactly $t = 3$ seconds, we can discover what Wile E.'s velocity is approaching at $t = 3$ seconds.

So, what is a limit?

The limit of a function is the *y*-value that the function approaches, as *x* gets closer and closer to a specific value.

Example: Sketch the graph of $f\left(x\right)=2x-3$.



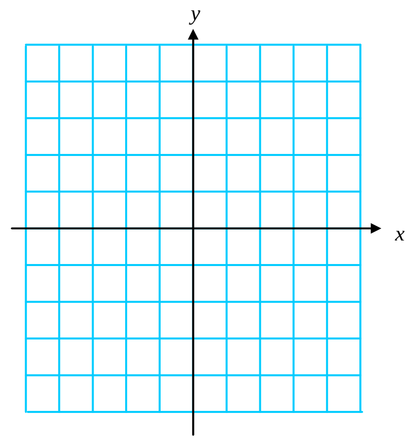
a) What is $f(2)$?

b) Complete the table of values below to determine what happens as x gets "close" to 2.

x approaches 2 from the **left** ⟶ | ⟵ x approaches 2 from the **right**

x	1.5	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25	2.5
$f(x)$											

Example: Sketch the graph of $f\left(x\right)=\dfrac{x^2-4}{x-2}$.



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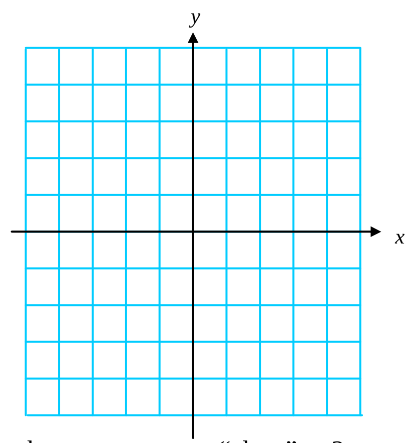
Example: Sketch the graph of $f\left(x\right)=\dfrac{1}{x-2}+3$.



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Definition of a Limit as x approaches a

As x approaches a, the **limit** of $f(x)$ is L, written

$$\lim_{x\to a} f(x) = L$$

if all the values of $f(x)$ get "close" to L when x is sufficiently close, but not necessarily equal to, a.

Example: Using the graphs and/or tables from the previous examples, find each limit.

a) $\lim_{x\to 2}(2x-3) =$

b) $\lim_{x\to 2}\dfrac{x^2-4}{x-2} =$

c) $\lim_{x\to 2}\left(\dfrac{1}{x-2}+3\right) =$

Look back at the last example on the last page page. As x approached 2 from the left, the function went a different direction than when x approached 2 from the right.

When x approaches 2 from the left, we use the notation __________________.

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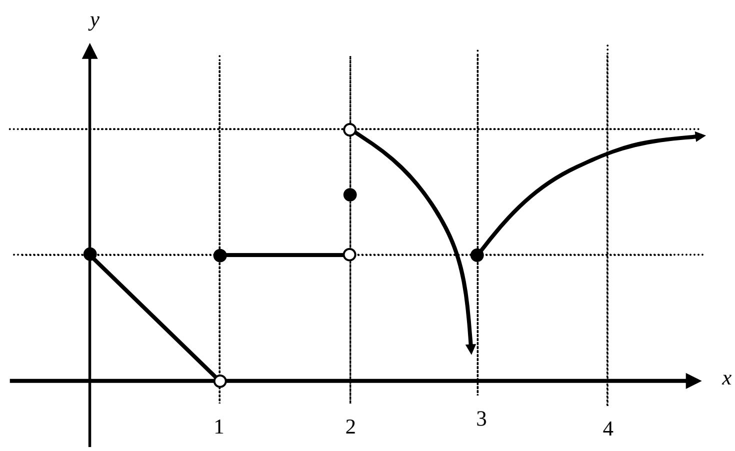
Using this notation, we can evaluate limits from the right or from the left individually.



Example: Using the graph above, evaluate each limit.

a) $\lim_{x\to 0^+} f(x) =$

b) $\lim_{x\to 1^-} f(x) =$

c) $\lim_{x\to 1^+} f(x) =$

d) $\lim_{x\to 2^-} f(x) =$

e) $\lim_{x\to 2^+} f(x) =$

f) $\lim_{x\to 3^-} f(x) =$

g) $\lim_{x\to 4^-} f(x) =$

h) $\lim_{x\to 4^+} f(x) =$

i) $\lim_{x\to \infty} f(x) =$

When do Limits Not Exist?

Limits fail for a variety of reasons, all of which can be traced back to the definition.

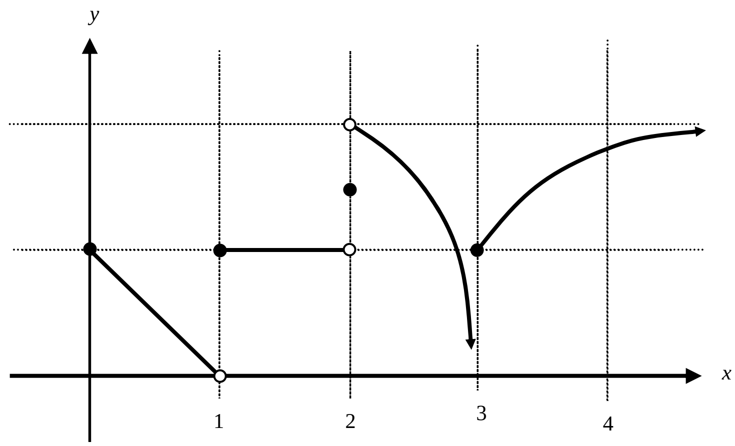
The definition states that the limit as x approaches a is L. But L must be a real number …

#1: Limits fail when the function approaches infinity as x approaches a.

When we built our tables, we looked at values of the function as x approaches a FROM BOTH SIDES.

#2: Limits fail when the function approaches different values from the right and left.

Example: Let's look at the last picture again.



a) $\lim_{x\to 0} f(x) =$

b) $\lim_{x\to 1} f(x) =$

c) $\lim_{x\to 2} f(x) =$

d) $\lim_{x\to 3} f(x) =$

e) $\lim_{x\to 4} f(x) =$

Example: Consider the function $g(x) = \begin{cases} 2x-3 & ; x>1 \\ 5-2x & ; x\le 1 \end{cases}$. Find $\lim_{x\to 1^+} g(x)$. Find $\lim_{x\to 1^-} g(x)$. What about $\lim_{x\to 1} g(x)$?