not a KEY

AP Calculus
6.2 Worksheet

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

Indefinite Integrals (Straight substitution)

1. $\int \frac{2x}{\sqrt{x^2+6}}\,dx$ $\quad \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = 2u^{1/2} + C$ $\quad \boxed{= 2\sqrt{x^2+6} + C}$

Let $u = x^2+6$
$du = 2x\,dx$

2. $\int \frac{e^x}{e^x+4}\,dx$ $\quad \int \frac{du}{u} = \ln|u| + c$ $\quad \boxed{= \ln(e^x+4) + c}$

Let $u = e^x+4$
$du = e^x\,dx$

Definite Integrals (straight substitution)

3. $\int_1^{\sqrt{2}} x\cdot 2^{-x^2}\,dx$ $\quad -\frac{1}{2}\int_{-1}^{-2} 2^u\,du$ $\quad -\frac{1}{2}\cdot \frac{2^u}{\ln 2}\Big|_{-1}^{-2}$

Let $u = -x^2$
$du = -2x\,dx \Rightarrow -\frac{1}{2}\,du = x\,dx$

if $x = \sqrt{2}$, $u = -(\sqrt{2})^2 = -2$
$x = 1$, $u = -(1)^2 = -1$

$\frac{-1\cdot 2^{(-2)}}{2\ln 2} - \frac{-1\cdot 2^{(-1)}}{2\ln 2}$

$\frac{-1}{8\ln 2} + \frac{1}{4\ln 2} = \boxed{\frac{1}{8\ln 2}}$

4. $\int_e^{e^2} \frac{1}{x\ln x}\,dx$ $\quad \int_1^2 \frac{1}{u}\,du = \ln|u|\Big|_1^2 = \ln(2) - \ln(1) = \boxed{\ln 2}$

Let $u = \ln x$
$du = \frac{1}{x}\,dx$

if $x = e^2$, $u = \ln e^2 = 2$
if $x = e$, $u = \ln e = 1$

5. True or $\boxed{\text{False}}$: $\int_0^{\pi/4} \tan^3(x)\sec^2(x)\,dx = \int_0^{\pi/4} u^3\,du$

Let $u = \tan(x)$
$du = \sec^2(x)\,dx$

↳ Limit's should be $\int_0^1$ (u-values ...) since if $x = \pi/4$, $u = \tan(\pi/4) = 1$; if $x = 0$, $u = \tan(0) = 0$

Algebraic Techniques

6. $\int \frac{e^x+4}{e^x}\,dx = \int\left(\frac{e^x}{e^x} + \frac{4}{e^x}\right)dx$

$= \int (1 + 4e^{-x})\,dx$

$= x + 4\cdot\frac{e^{-x}}{-1} + c$

$\boxed{= x - \frac{4}{e^x} + C}$

7. $\int_0^1 \frac{3\,dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{3\,dx}{(x+1)\sqrt{x^2+2x+1-1}}$

Complete the square

$\frac{1}{2}(2) = 1$
$1^2 = 1$

$= \int \frac{3\,dx}{(x+1)\sqrt{(x+1)^2-1}}$

Let $u = x+1$
$du = dx$

$= 3\int \frac{du}{u\sqrt{u^2-1}}$

$= 3\sec^{-1}|u| + C$

$\boxed{= 3\sec^{-1}|x+1| + C}$

8. $\int \frac{dx}{x^2-4x+4} = \int \frac{dx}{x^2-4x+4+4-4}$

Already a perfect square! $\quad \frac{1}{2}(-4)=-2 \quad (-2)^2=4$

$$\int \frac{dx}{(x-2)^2} = \int \frac{du}{u^2} = \int u^{-2}\,du$$

Let $u=x-2$

$du=dx$

$$= \frac{u^{-1}}{-1} + C$$

$$= -\frac{1}{u} + C$$

$$\boxed{= \frac{-1}{x-2} + C}$$

9. $\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$

$$= \int \frac{x}{2x\sqrt{x-1}}\,dx + \int \frac{\cancel{2\sqrt{x-1}}}{\cancel{2}x\cancel{\sqrt{x-1}}}\,dx$$

$$= \int \frac{dx}{2\sqrt{x-1}} + \int \frac{1}{x}\,dx$$

Let $u=x-1$

$du=dx$

$$= \frac{1}{2}\int \frac{du}{\sqrt{u}} + \ln|x| + C$$

$$= \frac{1}{2}\int u^{-1/2}\,du + \ln|x| + C$$

$$= \frac{1}{2}2u^{1/2} + \ln|x| + C = \boxed{\sqrt{x-1} + \ln|x| + C}$$

10. $\int \frac{dx}{\sqrt{-x^2+4x-3}}$

$$\int \frac{dx}{\sqrt{-1(x^2-4x+4)-3+4}}$$

$\frac{1}{2}(-4)=-2$

$(-2)^2=4$

$$\int \frac{dx}{\sqrt{1-(x-2)^2}} = \int \frac{du}{\sqrt{1-u^2}}$$

Let $u=(x-2)$

$du=dx$

$$= \sin^{-1}(u) + C$$

$$\boxed{= \sin^{-1}(x-2) + C}$$

11. $\int \frac{x^5-35x}{x^2+6}\,dx$

$$\begin{array}{r}
x^3 - 6x \qquad\qquad\qquad \\
x^2+0x+6 \enclose{longdiv}{x^5+0x^4+0x^3+0x^2-35x+0} \\
-(x^5+0x^4+6x^3) \qquad\qquad \\
-6x^3+0x^2-35x \\
-(-6x^3+0x^2-36x) \\
x
\end{array}$$

$$\int \left(x^3-6x+\frac{x}{x^2+6}\right)dx$$

$$\frac{1}{4}x^4-3x^2+\int \frac{x}{x^2+6}\,dx$$

Let $u=x^2+6$

$du=2x\,dx$

$\frac{1}{2}du=x\,dx$

$$\frac{1}{4}x^4-3x^2+\frac{1}{2}\int \frac{du}{u}$$

$$\frac{1}{4}x^4-3x^2+\frac{1}{2}\ln|u|+C$$

$$\boxed{\frac{1}{4}x^4-3x^2+\frac{1}{2}\ln(x^2+6)+C}$$

Inv Trig Function examples

12. $\int \frac{dx}{2+9x^2} = \int \frac{dx}{2\left(1+\frac{9x^2}{2}\right)} = \frac{1}{2}\int \frac{dx}{1+\frac{9x^2}{2}}$

$u=\frac{3x}{\sqrt{2}}$

$du=\frac{3}{\sqrt{2}}dx \Rightarrow \frac{\sqrt{2}}{3}du=dx$

$$= \frac{1}{2}\int \frac{\frac{\sqrt{2}}{3}du}{1+u^2}$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{3}\int \frac{du}{1+u^2}$$

$$= \frac{\sqrt{2}}{6}\tan^{-1}(u) + C$$

$$\boxed{= \frac{\sqrt{2}}{6}\tan^{-1}\left(\frac{3x}{\sqrt{2}}\right) + C}$$

13. $\int \frac{dx}{\sqrt{e^{2x}-1}}$

$\boxed{\text{Let } u=e^x}$

$du=e^x\,dx \Rightarrow \boxed{\frac{1}{e^x}du=dx} \Rightarrow$

$$\int \frac{\frac{1}{e^x}du}{\sqrt{u^2-1}} = \int \frac{\frac{1}{u}du}{\sqrt{u^2-1}}$$

$$= \int \frac{du}{u\sqrt{u^2-1}}$$

$$= \sec^{-1}(u) + C$$

$$\boxed{= \sec^{-1}(e^x) + C}$$

(14) $\displaystyle\int \frac{e^x}{1+2e^x}\,dx$

Let $u = 1+2e^x$

$du = 2e^x\,dx$

$\Rightarrow \frac{1}{2}du = e^x\,dx$

$$\int \frac{\frac{1}{2}du}{u} = \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C$$

$$\boxed{= \frac{1}{2}\ln|1+2e^x| + C}$$

(15) $\displaystyle\int \sec^2(2x)\,dx$

Let $u = 2x$

$du = 2\,dx$

$\Rightarrow \frac{1}{2}du = dx$

$$= \frac{1}{2}\int \sec^2(u)\,du$$

$$= \frac{1}{2}\tan(u) + C$$

$$\boxed{= \frac{1}{2}\tan(2x) + C}$$

(16) $\displaystyle\int \sec^2(3x)\,e^{\tan(3x)}\,dx =$

Let $u = \tan(3x)$

$du = \sec^2(3x)\cdot 3\,dx$

$\Rightarrow \frac{1}{3}du = \sec^2(3x)\,dx$

$$= \frac{1}{3}\int e^u\,du$$

$$= \frac{1}{3}e^u + C$$

$$\boxed{= \frac{1}{3}e^{\tan(3x)} + C}$$

(17) $\displaystyle\int \frac{x}{2x^2+1}\,dx$

Let $u = 2x^2+1$

$du = 4x\,dx$

$\Rightarrow \frac{1}{4}du = x\,dx$

$$\frac{1}{4}\int \frac{du}{u} = \frac{1}{4}\ln|u| + C$$

$$\boxed{= \frac{1}{4}\ln|2x^2+1| + C}$$

(18) $\displaystyle\int e^x(2+e^x)^{1/2}\,dx$

Let $u = 2+e^x$

$du = e^x\,dx$

$$= \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C$$

$$\boxed{= \frac{2}{3}(2+e^x)^{3/2} + C}$$

(19) $\int x^2 \cos(x^3)\,dx$

Let $u = x^3$

$du = 3x^2 dx$

$\Rightarrow \frac{1}{3} du = x^2 dx$

$= \frac{1}{3}\int \cos(u)\,du$

$= \frac{1}{3}\sin(u) + C$

$\boxed{= \frac{1}{3}\sin(x^3) + C}$

(20) $\int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du$

$u = \tan x$

$du = \sec^2 x\,dx$

$= 2u^{1/2} + C$

$\boxed{= 2\sqrt{\tan x} + C}$

(21) $\int \frac{\tan^{-1} x}{1+x^2}\,dx = \int u\,du = \frac{1}{2}u^2 + C$

$\boxed{= \frac{1}{2}(\tan^{-1}x)^2 + C}$

Let $u = \tan^{-1} x$

$du = \frac{dx}{1+x^2}$

(22) $\int \csc^2(3x+5)\,dx$

$u = 3x+5$

$du = 3dx$

$\Rightarrow \frac{1}{3} du = dx$

$= \frac{1}{3}\int \csc^2(u)\,du$

$= \frac{1}{3}(-\cot(u)) + C$

$\boxed{= -\frac{1}{3}\cot(3x+5) + C}$

(23) $\int \frac{x+1}{(x^2+2x+7)^3}\,dx$

$u = x^2+2x+7$

$du = (2x+2)\,dx$

$\Rightarrow \frac{1}{2} du = (x+1)\,dx$

$\frac{1}{2}\int \frac{du}{u^3} = \frac{1}{2}\int u^{-3}\,du$

$= \frac{1}{2} \cdot \frac{u^{-2}}{-2} + C$

$\boxed{= -\frac{1}{4}(x^2+2x+7)^{-2} + C}$

(24) $\int \frac{x}{x^2-4}\,dx$

Let $u = x^2-4$

$du = 2x\,dx$

$\Rightarrow \frac{1}{2}du = x\,dx$

$$= \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C$$

$$\boxed{= \frac{1}{2}\ln|x^2-4| + C}$$

(25) $\int x \tan^2(x^2)\,dx$

Let $u = x^2$

$du = 2x\,dx$

$\Rightarrow \frac{1}{2}du = x\,dx$

$$= \frac{1}{2}\int \tan^2(u)\,du$$

$$= \frac{1}{2}\left[\tan(u) - u\right] + C$$

$$\boxed{= \frac{1}{2}\tan(x^2) - \frac{1}{2}x^2 + C}$$

(26) $\int \cos(3x)\,e^{\sin(3x)}\,dx$

Let $u = \sin(3x)$

$du = \cos(3x)\cdot 3\,dx$

$\Rightarrow \frac{1}{3}du = \cos(3x)\,dx$

$$= \frac{1}{3}\int e^u\,du$$

$$= \frac{1}{3}e^u + C$$

$$\boxed{= \frac{1}{3}e^{\sin(3x)} + C}$$

(27) $\int \frac{1}{x\ln(3x)}\,dx$

Let $u = \ln(3x)$

$du = \frac{3}{3x}dx = \frac{dx}{x}$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$\boxed{= \ln|\ln(3x)| + C}$$

(28) $\displaystyle\int \frac{\sin(3x)}{1+\cos(3x)}\,dx$

Let $u = 1+\cos(3x)$

$du = -\sin(3x)\cdot 3\,dx$

$\Rightarrow -\frac{1}{3}du = \sin(3x)dx$

$$= -\frac{1}{3}\int \frac{du}{u}$$

$$= -\frac{1}{3}\ln|u| + C$$

$$\boxed{= -\frac{1}{3}\ln\left|1+\cos(3x)\right| + C}$$

(29) $\displaystyle\int \frac{1}{x^2-2x+17}\,dx = \int \frac{dx}{x^2-2x+\underline{1}+17-\underline{1}} = \int \frac{dx}{(x-1)^2+16}$

$\frac{1}{2}(-2) = -1$

$(-1)^2 = 1$

$$= \frac{1}{16}\int \frac{dx}{\frac{(x-1)^2}{16}+1}$$

Let $u = \left(\frac{x-1}{4}\right)$

$du = \frac{1}{4}dx$

$\Rightarrow 4du = dx$

$$= \frac{1}{16}\int \frac{4du}{u^2+1}$$

$$= \frac{1}{4}\tan^{-1}(u) + C$$

$$\boxed{= \frac{1}{4}\tan^{-1}\left(\frac{x-1}{4}\right) + C}$$

(30) $\displaystyle\int \frac{1}{\sqrt{1-9x^2}}\,dx = \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}}$

Let $u = 3x$

$du = 3dx$

$\Rightarrow \frac{1}{3}du = dx$

$$= \frac{1}{3}\sin^{-1}(u) + C$$

$$\boxed{= \frac{1}{3}\sin^{-1}(3x) + C}$$

(31) $\int x \csc(3x^2)\cot(3x^2)\,dx$

Let $u = 3x^2$

$du = 6x\,dx \Rightarrow \frac{1}{6}du = x\,dx$

$$= \frac{1}{6}\int \csc(u)\cot(u)\,du$$

$$= -\frac{1}{6}\csc(u) + C$$

$$= \boxed{-\frac{1}{6}\csc(3x^2) + C}$$

(32) $\int \frac{1 - e^{-x}}{x + e^{-x}}\,dx$

Let $u = x + e^{-x}$

$du = (1 - e^{-x})\,dx$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$\boxed{= \ln|x + e^{-x}| + C}$$

(33) $\int \frac{x^2-1}{x^2+1}\,dx = \int \left(1 + \frac{-2}{x^2+1}\right)dx$

$$\boxed{= x - 2\tan^{-1}(x^2+1) + C}$$

$$\begin{array}{r|l} & 1 \\ x^2+0x+1 & x^2+0x-1 \\ & -(x^2+0x+1) \\ \hline & -2 \end{array}$$

★★ (34) $\int (x+1)\sqrt{2-x}\,dx$

Let $u = 2-x \Rightarrow x = 2-u \ldots$

$du = -dx$

$\Rightarrow -du = dx$

$$= -\int [(2-u)+1]\sqrt{u}\,du$$

$$= -\int (3-u)u^{1/2}\,du$$

$$= -\int (3u^{1/2} - u^{3/2})\,du$$

$$= -\left[3\cdot\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right] + C$$

$$\boxed{= -2(2-x)^{3/2} - \frac{2}{5}(2-x)^{5/2} + C}$$

(35) $\int \frac{x+2}{\sqrt{4-x^2}}\,dx = \int \frac{x}{\sqrt{4-x^2}}\,dx + \int \frac{2}{\sqrt{4-x^2}}\,dx$

Let $u = 4-x^2$

$du = -2x\,dx$

$\Rightarrow -\frac{1}{2}du = x\,dx$

$$= 2\int \frac{dx}{\sqrt{4(1-x^2/4)}}$$

$$= \int \frac{dx}{\sqrt{1-\frac{x^2}{4}}}$$

Let $w = \frac{x}{2}$ & $dw = \frac{1}{2}dx$

$\therefore w^2 = \frac{x^2}{4}$

$\Rightarrow 2dw = dx$

$$= -\frac{1}{2}\int \frac{du}{\sqrt{u}} + \int \frac{2\,dw}{\sqrt{1-w^2}}$$

$$= -\frac{1}{2}\int u^{-1/2}\,du + 2\int \frac{dw}{\sqrt{1-w^2}}$$

$$= -\frac{1}{2}\cdot 2u^{1/2} + 2\sin^{-1}(w) + C$$

$$\boxed{= -\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right) + C}$$

(36) $\int_0^2 \sqrt{4x+1}\,dx$

Let $u = 4x+1$

$du = 4dx \Rightarrow \frac{1}{4}du = dx$

$x=2 \Rightarrow u=9$

$x=0 \Rightarrow u=1$

$$= \frac{1}{4}\int_1^9 \sqrt{u}\,du$$

$$= \frac{1}{4}\int_1^9 u^{1/2}\,du$$

$$= \frac{1}{4}\cdot\frac{2}{3}u^{3/2}\Big|_1^9$$

$$= \frac{1}{6}u^{3/2}\Big|_1^9$$

$$= \frac{1}{6}(9)^{3/2} - \frac{1}{6}(1)^{3/2}$$

$$= \frac{1}{6}\cdot 27 - \frac{1}{6}$$

$$= \frac{26}{6}$$

$$\boxed{= 13/3}$$

(37) $\int_{-1}^{1} \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_{-1}^{1} = \tan^{-1}(1) - \tan^{-1}(-1) = \pi/4 - (-\pi/4) = \boxed{\pi/2}$

NO SUBSTITUTION REQUIRED

(38) $\int_0^1 \frac{1}{(2x+3)^3}\,dx$

Let $u = 2x+3$

$du = 2dx \Rightarrow \frac{1}{2}du = dx$

$x=1 \Rightarrow u=5$

$x=0 \Rightarrow u=3$

$$= \frac{1}{2}\int_3^5 u^{-3}\,du = \frac{1}{2}\cdot\frac{1}{-2}u^{-2}\Big|_3^5$$

$$= -\frac{1}{4}u^{-2}\Big|_3^5$$

$$= -\frac{1}{4}(5)^{-2} - \left[-\frac{1}{4}(3)^{-2}\right]$$

$$= -\frac{1}{4}\cdot\frac{1}{25} + \frac{1}{4}\cdot\frac{1}{9}$$

$$= \frac{-1}{100} + \frac{1}{36}$$

$$= \frac{-9+25}{900}$$

$$= \frac{16}{900} = \boxed{\frac{4}{225}}$$

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$$\int_2^{e+1} \frac{x}{(x-1)^2}\,dx$$

Let $u = x-1 \Rightarrow u+1 = x$

$du = dx$

$x = e+1 \Rightarrow u = e$

$x = 2 \Rightarrow u = 1$

$$= \int_1^e \left(\frac{u+1}{u^2}\right) du = \int_1^e \left(\frac{u}{u^2} + \frac{1}{u^2}\right) du$$

$$= \int_1^e \left(\frac{1}{u} + u^{-2}\right) du$$

$$= \ln|u| + \frac{u^{-1}}{-1} \Big|_1^e$$

$$= \ln|u| - \frac{1}{u} \Big|_1^e$$

$$= \left(\ln|e| - \frac{1}{e}\right) - \left(\ln|1| - \frac{1}{1}\right)$$

$$= \left(1 - \frac{1}{e}\right) - (0 - 1)$$

$$= 1 - \frac{1}{e} + 1$$

$$\boxed{= 2 - \frac{1}{e}}$$

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$$\int_0^\pi \sin\left(\frac{x}{2}\right) dx = \int_0^\pi \sin\left(\frac{1}{2}x\right) dx$$

$$= \frac{-\cos\left(\frac{1}{2}x\right)}{1/2} \Bigg|_0^\pi = \left[-2\cos\left(\tfrac{1}{2}\cdot\pi\right)\right] - \left[-2\cos\left(\tfrac{1}{2}\cdot 0\right)\right]$$

$$= -2\cdot 0 + 2\cdot 1$$

$$= \boxed{2}$$

(41) $\displaystyle\int_{-\pi}^{\pi} x \sin(x^2)\, dx$

Let $u = x^2$

$du = 2x\,dx \Rightarrow \frac{1}{2} du = x\,dx$

$x = \pi \Rightarrow u = \pi^2$

$x = -\pi \Rightarrow u = \pi^2$

$$= \frac{1}{2}\int_{\pi^2}^{\pi^2} \sin(u)\, du \ \boxed{= 0} \ \ldots$$ (same Limits!)

(42) $\displaystyle\int_{-1}^{0} \frac{2}{6x-1}\, dx$

Let $u = 6x-1$

$du = 6\,dx \Rightarrow \frac{1}{6} du = dx$

$x = 0 \Rightarrow u = -1$

$x = -1 \Rightarrow u = -7$

$$= \frac{2}{6}\int_{-7}^{-1} \frac{du}{u}$$

$$= \frac{1}{3} \ln|u| \Big|_{-7}^{-1}$$

$$= \frac{1}{3}\ln|-1| - \frac{1}{3}\ln|-7|$$

$$= 0 - \frac{1}{3}\ln(7)$$

$$\boxed{= -\frac{1}{3}\ln 7}$$

(43) $\displaystyle\int_{1}^{5} \frac{(\ln x)^{1/2}}{x}\, dx$

Let $u = \ln x$

$du = \frac{1}{x} dx$

$x = 5 \Rightarrow u = \ln 5$

$x = 1 \Rightarrow u = \ln 1 = 0$

$$= \int_{0}^{\ln 5} u^{1/2}\, du$$

$$= \frac{2}{3} u^{3/2} \Big|_0^{\ln 5}$$

$$= \frac{2}{3}(\ln 5)^{3/2} - \frac{2}{3}(0)^{3/2}$$

$$\boxed{= \frac{2}{3}(\ln 5)^{3/2}}$$

(44) $\displaystyle\int_{0}^{1} x e^{-x^2}\, dx$

Let $u = -x^2 \ldots du = -2x\,dx$

$\Rightarrow -\frac{1}{2} du = x\,dx$

$x = 1 \Rightarrow u = -1$

$x = 0 \Rightarrow u = 0$

$$= -\frac{1}{2}\int_0^{-1} e^u\, du$$

$$= -\frac{1}{2} e^u \Big|_0^{-1} = -\frac{1}{2}e^{-1} - \left(-\frac{1}{2}e^0\right)$$

$$= \boxed{-\frac{1}{2e} + \frac{1}{2}} = \boxed{\frac{e-1}{2e}}$$

(45) $\displaystyle\underbrace{\int_0^2 (2^x + x^2)\, dx}_{\text{NO SUBSTITUTION REQUIRED}} = \frac{2^x}{\ln 2} + \frac{1}{3}x^3 \Big|_0^2$

$$= \left(\frac{2^2}{\ln 2} + \frac{1}{3}(2)^3\right) - \left[\frac{2^0}{\ln 2} + \frac{1}{3}(0)^3\right]$$

$$= \frac{4}{\ln 2} + \frac{8}{3} - \frac{1}{\ln 2} - 0$$

$$= \boxed{\frac{3}{\ln 2} + \frac{8}{3}} = \boxed{\frac{9 + 8\ln 2}{3\ln 2}}$$