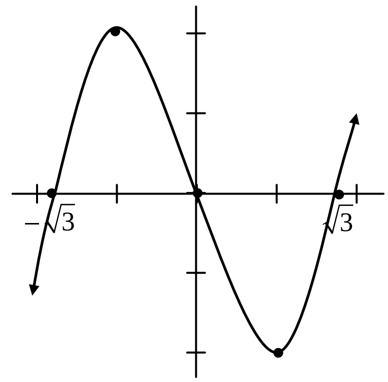
Calculus AB
Chapter 3.6 – 4.4 Exam Review Solutions

****Here's a quick list of answers … explanations follow …****

1. $\frac{1}{5}$
2. 4.509
3. $-\frac{1}{2}$
4. a) $x = 0$ or $x = \pm\sqrt{3}$
   b) y is increasing when $x < -1$ and when $x > 1$
   y is decreasing when $-1 < x < 1$
   c) MAX = 2
   MIN = –2
   d) y is concave up when $x > 0$
   y is concave down when $x < 0$
   There is a point of inflection at (0, 0).



Sketch of y …
WITHOUT
CALCULATOR!!!

5. $v(t) = 5t$
   $s(t) = \frac{5}{2}t^2 + 5$

6. a) $y' = \frac{2x}{\sqrt{1-x^4}}$

   b) $y' = \frac{\cos x}{1+\sin^2 x}$

   c) $\frac{\frac{1}{2}x^{-\frac{1}{2}}}{\sqrt{x}\sqrt{x-1}} = \frac{1}{2|x|\sqrt{x-1}}$

   d) $e^{-x^2} \cdot (-2x)$

   e) $5^{x^2+5} \cdot \ln(5) \cdot 2x$

   f) $\frac{2}{\sqrt{1-4x^2} \cdot 4^y \cdot \ln(4)} = \frac{2}{\sqrt{1-4x^2} \cdot \sin^{-1}(2x) \cdot \ln(4)}$

   g) $y' = e^x\left[\cot x + \ln(\sin x)\right] \cdot (\sin x)^{e^x}$

7. Concave up when $x > 4$
   Concave down when $x < 0$ and $0 < x < 4$
   Point of inflection when $x = 4$ only.

8. @ a … $g'(x) < 0$, since g is decreasing and $g''(x) > 0$ since g is concave up.
   @ b … $g'(x) = 0$, since g has a horizontal tangent line and $g''(x) > 0$ since g is concave up.
   @ c … $g'(x) = 0$, since g has a horizontal tangent line and $g''(x) < 0$ since g is concave down.
   @ d … $g'(x) < 0$, since g is decreasing and $g''(x) < 0$ since g is concave down.

9. f is increasing when $x < b$ and when $x > d$, since f' is positive
   f is decreasing when $b < x < d$, since f' is negative
   f is concave up when $x > c$, since f' is increasing
   f is concave down when $x < c$, since f' is decreasing
   f has a maximum when $x = b$, since f' changed from + to –
   f has a minimum when $x = d$, since f' changed from – to +

10. a) $y' = 3^{\sin x} \cdot \ln(3) \cdot \cos x$
    b) $y' = 1$

11. Extrema occur at critical points (points where $f\,'$ is undefined or equals 0) OR at the endpoints of a closed interval.

There are <u>two ways</u> to determine whether or not the extrema is a maximum or minimum.

#1: Using the first derivative test, you see if the first derivative changes sign at that critical point.

If the first derivative changes from + to –, there is a maximum at that critical point.
If the first derivative changes from – to +, there is a minimum at that critical point.

#2: Using the second derivative test, you plug the critical numbers into the second derivative.

If the value of the second derivative at the critical point is positive, then there is a minimum at that critical point, and if the value of the second derivative at the critical point is negative, then there is a maximum at that critical point.

Absolute extrema vs. Relative extrema:

An absolute maximum is the largest y-value in the entire interval.
An absolute minimum is the smallest y-value in the entire interval.
A relative maximum is a y-value that is larger than all the other y-values that are close to it.
A relative minimum is a y-value that is smaller than all the other y-values that are close to it.

12. $\frac{dy}{dx}=\frac{1-2xy}{x^2+6y}$

13. $y'''(x)=13440(4x+1)^4$

14. Maximum Area = 25 square units.

15. g is increasing on $(-2\pi,-3.665)\cup(-1.571,2.618)\cup(4.712,2\pi)$, since $g\,'$ is positive

g is decreasing on $(-3.665,-1.571)\cup(2.618,4.712)$, since $g\,'$ is negative

g is concave up on $(-2\pi,-5.760)\cup(-2.618,0.524)\cup(3.665,2\pi)$, since $g\,'$ is increasing

g is concave down on $(-5.760,-2.618)\cup(0.524,3.665)$, since $g\,'$ is decreasing

g has points of inflection when $x\approx-5.760$, $x\approx-2.618$, $x\approx0.524$, and $x\approx3.665$, since the concavity of g changes when $g\,'$ changes from increasing to decreasing or from increasing to decreasing.

****Explanations/Work****

1. Since the derivative of the inverse function is the reciprocal of the derivative of the function at the inverse point, we must find what values of x give 2 for the original function. In other words,

$$\left(f^{-1}\right)'(2)=\frac{1}{f\,'(?)}$$, and we need to find the value of "?".

Solve the equation $2=x^3+2x-1$ for x by setting the equation equal to zero, and using your calculator to solve for the zeros. You get x = 1. Since $f\,'(x)=3x^2+2$, we get

$$\left(f^{-1}\right)'(2)=\frac{1}{f\,'(1)}=\frac{1}{3(1)^2+2}=\frac{1}{5}$$

2. The MVT states that as long as the function is continuous on the closed interval [*a*, *b*] and differentiable on the open interval (*a*, *b*), then there is a point *c* between *a* and *b*, such that v … (the derivative at *c* is the same as the slope of the line containing the endpoints). So, since $f'(x) = \cos x$, we need a value of *c* where

$$\cos c = \frac{\sin 5 - \sin 4}{5-4}$$

Solve this equation for *c*:

$$\cos c \approx -.2021217794$$
$$c \approx \cos^{-1}(-.2021217794)$$
$$c \approx 1.77432026$$

Note, however, that this value of *c* is NOT in the given interval. This value of *c* is in the second quadrant. Since cosine is also negative in the third quadrant, we just need to find out what value in the third quadrant corresponds to 1.77432026. Since the reference angle would have a value of 1.367272393, the value in the third quadrant with the same reference angle would be 4.508865047. Notice, this number IS in the given interval. Therefore, the value of *c* we are looking for is 4.509.

***Since this is a calculator question you could have solved cos *c* = –.2021217794 graphically … just find the solution that falls within the given interval.

3. Using the interval [–2, 1], we have to solve for *c* using the MVT $\left[f'(c) = \frac{f(b) - f(a)}{b-a}\right]$.

$$8c + 5 = \frac{9-(6)}{1-(-2)}$$
$$8c + 5 = 1$$
$$c = -\tfrac{1}{2}$$

Notice, that *c* IS in the interval!

4. You should try to do this problem completely without a calculator …

a) To find the zeros, factor out an *x* from the function … $y = x(x^2 - 3)$ … so *y* = 0, when $x = 0$ or $x = \pm\sqrt{3}$.

b) To determine increasing/decreasing find *y* ' … $y' = 3x^2 - 3$ … *y* ' is never undefined, and *y* ' = 0, when $x = \pm 1$. Use a sign chart with *y* ' (or a graph of *y* ') and these 2 critical numbers.

Since *y* ' is positive, *y* is increasing when *x* < –1 and when *x* > 1.
Since *y* ' is negative, *y* is decreasing when –1 < *x* < 1.

c) Since *y* ' > 0 on (–∞, –1) and *y* ' < 0 on (–1, 1), *y* has a maximum when *x* = –1. MAX = $(-1)^3 - 3(-1) = 2$.

Since *y* ' < 0 on (–1, 1) and *y* ' > 0 on (1, ∞), *y* has a minimum when *x* = 1. MIN = $(1)^3 - 3(1) = -2$.

d) To determine concavity, find *y* " … *y* " = 6*x* … *y* " is never undefined, and *y* " = 0 when *x* = 0. Use a sign chart with *y* " (or a graph of *y* ") .

Since *y* " is positive, *y* is concave up when *x* > 0.
Since *y* " is negative, *y* is concave down when *x* < 0.
Since *y* " changed signs (or *y* changed concavity) at *x* = 0, there is a point of inflection at (0, 0).

Using all of this information above … you should be able to SKETCH the function … include the zeros … the max and min points the point of inflection, and then draw the graph so that it is increasing and decreasing on the appropriate intervals and the concavity is appropriate to it's intervals.

(see sketch on first page)

5. Since *a* (*t*) = 5, then *v* (*t*) = 5*t* + *C*, where *C* is some constant.
Use the fact that *v* (2) = 10, to get 10 = 5(2) + *C*, which means *C* = 0 and *v* (*t*) = 5*t*.
Since *v* (*t*) = 5*t*, then $s(t) = \frac{5}{2}t^2 + K$, where *K* is some constant.

Use the fact that *s* (0) = 5, to get $5 = \frac{5}{2}(0)^2 + K$, which means *K* = 5 and $s(t) = \frac{5}{2}t^2 + 5$.

6. a) Using the rule for the derivative of inverse sine … $y' = \frac{2x}{\sqrt{1-x^4}}$.

b) Using the rule for the derivative of inverse tangent … $y' = \frac{\cos x}{1+\sin^2 x}$

c) Using the rule for the derivative of inverse secant … $\frac{\frac{1}{2}x^{-\frac{1}{2}}}{\sqrt{x}\sqrt{x-1}}$. Writing this as one fraction with positive exponents gives us

$$\frac{1}{2\sqrt{x}\sqrt{x}\sqrt{x-1}}$$

Since $\sqrt{x}\sqrt{x} = \sqrt{x^2} = |x|$, we could write the answer as

$$\frac{1}{2|x|\sqrt{x-1}}.$$

d) Using the rule for the derivative of e^u … $e^{-x^2} \cdot (-2x)$

e) Using the rule $a^u = a^u \cdot \ln(a) \cdot u'$ (where *a* is a constant and *u* is a differentiable function) … $5^{x^2+5} \cdot \ln(5) \cdot 2x$

f) …. Two ways …

Using the rules of logarithms, we can rewrite this to be

$$4^y = \sin^{-1}(2x).$$

Using implicit differentiation, we have

$$4^y \cdot \ln(4) \cdot \frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{2}{\sqrt{1-4x^2} \cdot 4^y \cdot \ln(4)}$$

Since $4^y = \sin^{-1}(2x)$, we could write

$$\frac{2}{\sqrt{1-4x^2} \cdot \sin^{-1}(2x) \cdot \ln(4)}$$

How about a different way? ….

Use the change of base formula to write the original equation as

$$y = \frac{\ln\left(\sin^{-1}(2x)\right)}{\ln 4}$$

Since ln 4 is just a constant you can write the equation as

$$y = \frac{1}{\ln 4} \cdot \ln\left(\sin^{-1}(2x)\right).$$

Taking the derivative, we leave the constant out front and take the derivative using the derivative rules for ln *u* and the inverse sine derivative rules to get

$$y = \frac{1}{\ln 4} \cdot \left(\frac{2/\sqrt{1-4x^2}}{\sin^{-1}(2x)} \right)$$

Technically speaking, this is the answer for the derivative … you could (if you wanted) rearrange the fractions to obtain the answer we had in the first option.

g) NOTE: This is not like parts *d* and *e* where there was a numerical base raised to a functional power … This is a function raised to a functional power … meaning … logarithmic differentiation is required.

Using logarithmic differentiation, we can turn $y = (\sin x)^{e^x}$ into

$$\ln y = e^x \cdot \ln(\sin x)$$

Using implicit differentiation, we have

$$\frac{y'}{y} = e^x \cdot \frac{\cos x}{\sin x} + \ln(\sin x) \cdot e^x .$$

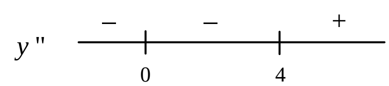
Factoring out an e^x on the right side, and multiplying both sides by *y*, we have

$$y' = e^x \left[\frac{\cos x}{\sin x} + \ln(\sin x) \right] \cdot y$$

Since $y = (\sin x)^{e^x}$, we can write

$$y' = e^x \left[\cot x + \ln(\sin x) \right] \cdot (\sin x)^{e^x}$$

7. If you factor out an x^2, you get $y'' = x^2(x-4)$. Since *y* " is never undefined, and *y* " = 0 when *x* = 0 and *x* = 4, we can use a sign chart (or a graph if you know what it should look like).
y is concave down when *x* < 0 and when 0 < *x* < 4 since *y* " is negative.
y is concave up, when *x* > 4, since *y* " is positive.



Since *y* " changes sign (or *y* changes concavity) ONLY at *x* = 4, there is a point of inflection when *x* = 4.
NOTICE!!! THERE IS ***NOT*** A POINT OF INFLECTION AT *x* = 0 SINCE THE CONCAVITY DOESN'T CHANGE!!!!!!

8. Since you are given a graph of *g*, EVERY ONE of your answers should be justified with … "since *g* …. "

@ *a* … *g* ' (*x*) < 0, since *g* is decreasing and *g* " (*x*) > 0 since *g* is concave up.
@ *b* … *g* ' (*x*) = 0, since *g* has a horizontal tangent line and *g* " (*x*) > 0 since *g* is concave up.
@ *c* … *g* ' (*x*) = 0, since *g* has a horizontal tangent line and *g* " (*x*) < 0 since *g* is concave down.
@ *d* … *g* ' (*x*) < 0, since *g* is decreasing and *g* " (*x*) < 0 since *g* is concave down.

9. Since you are given a graph of *f* ', EVERY ONE of your answers should be justified with … "since *f* ' … "

Since *f* ' is positive, *f* is increasing when *x* < *b* and when *x* > *d*
Since *f* ' is negative, *f* is decreasing when *b* < *x* < *d*
Since *f* ' is increasing, *f* is concave up when *x* > *c*
Since *f* ' is decreasing, *f* is concave down when *x* < *c*
Since *f* ' changed from + to – at *x* = *b*, *f* has a maximum when *x* = *b*
Since *f* ' changed from – to + at *x* = *b*, *f* has a minimum when *x* = *d*

10. a) Using the rule $a^u = a^u \cdot \ln(a) \cdot u'$ (where *a* is a constant and *u* is a differentiable function) … $3^{\sin x} \cdot \ln(3) \cdot \cos x$
b) Since the original function can be simplified to $y = x$ as long as *x* > 0 … *y* ' = 1 .

11. The answer given before IS the explanation as well.

12. Using implicit differentiation and simplifying we have,

$$\left[x^2 \frac{dy}{dx} + y \cdot 2x \right] + 6y \frac{dy}{dx} = 1$$

$$x^2 \frac{dy}{dx} + 6y \frac{dy}{dx} = 1 - 2xy$$

$$\left(x^2 + 6y \right) \frac{dy}{dx} = 1 - 2xy$$

$$\frac{dy}{dx} = \frac{1 - 2xy}{x^2 + 6y}$$

13. Using the chain rule (a few times) …

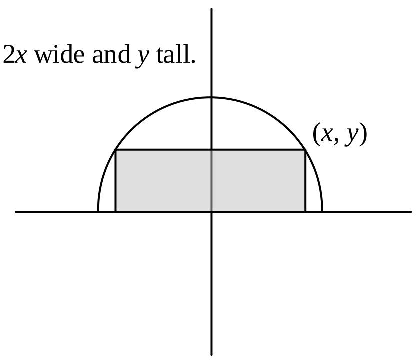
$$y' = 7(4x+1)^6 \cdot 4$$
$$= 28(4x+1)^6$$

$$y'' = 168(4x+1)^5 \cdot 4$$
$$= 672(4x+1)^5$$

$$y''' = 3360(4x+1)^4 \cdot 4$$
$$= 13440(4x+1)^4$$

14. A picture might help … notice *h* (*x*) is just the top half of a circle with radius 5.

If we label the point at the top right corner of the rectangle (*x*, *y*), then the rectangle is 2*x* wide and *y* tall.



Thus, the area we want to maximize is given by $A = 2xy$.
Since $y = \sqrt{25 - x^2}$, we really need to maximize the function $A = 2x\sqrt{25 - x^2}$.

Maximize *A* by finding *A* ': $A' = 2x\left[\frac{1}{2}\left(25 - x^2\right)^{-1/2} \cdot (-2x)\right] + \sqrt{25 - x^2} \cdot 2$

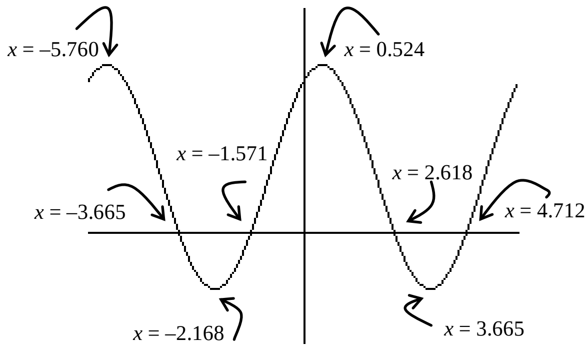
$$= \frac{-2x^2}{\sqrt{25 - x^2}} + 2\sqrt{25 - x^2}$$

Get a common denominator and simplify to obtain

$$A' = \frac{-2x^2 + 2\left(25 - x^2\right)}{\sqrt{25 - x^2}}$$
$$= \frac{50 - 4x^2}{\sqrt{25 - x^2}}$$

A ' is undefined when $x = \pm 5$, but these are the endpoints of your interval and would yield an area = 0.
A ' = 0, when $x = \pm\sqrt{12.5} \approx 3.535533906$. The way we've labeled our picture enables us to only use the positive value.
When $x = \sqrt{12.5}$, *A* = 25. To justify that this is a maximum value using the first derivative test, we could use a sign chart to determine that when $x = \sqrt{12.5}$, *A* ' changes from + to – , giving us a maximum area of 25.

15. Graphing the function g' and using your calculator to find all the following x-values, we can use this picture to determine the answers to all the questions. Since we are given a graph of g', EVERY ONE of your answers must be justified with …. "since g' …. "



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g is decreasing on $(-3.665,-1.571)\cup(2.618,4.712)$, since g' is negative

g is concave up on $(-2\pi,-5.760)\cup(-2.618,0.524)\cup(3.665,2\pi)$, since g' is increasing

g is concave down on $(-5.760,-2.618)\cup(0.524,3.665)$, since g' is decreasing

g has points of inflection when $x\approx-5.760$, $x\approx-2.618$, $x\approx 0.524$, and $x\approx 3.665$, since the concavity of g changes when g' changes from increasing to decreasing or from increasing to decreasing.