3.6 THE CHAIN RULE

Notecards from Section 3.6: The Chain Rule

Suppose you were asked to differentiate $h(x)=\sqrt{x^2+1}$.

Up to this point in the course, we have no tools with which to differentiate this function because there is a function $\left(x^2+1\right)$ inside another function $\sqrt{x}$, (aka a composite function). If we were to let

$$f(u)=\sqrt{u}\text{, and } u=g(x)=x^2+1,$$

then $h(x)=f\left(g(x)\right)=\sqrt{x^2+1}$. We can differentiate each of these separately, however, we need a rule allowing us to differentiate the original function, or more generally any composite function. Let us consider an example that may shed some light on how this might be accomplished.

Example 1: The length, *L*, in cm, of a steel bar depends on the air temperature, $H\ {}^\circ C$, and the temperature *H* depends on time, *t*, measured in hours. If the length increases by 2 cm for every degree increase in temperature, and the temperature is increasing at $3\ {}^\circ C$ per hour, how fast is the length of the bar increasing? What are the units for your answer?

This last example illustrates a practical look at how the following theorem actually works.

<u>*The Chain Rule*</u>

If $y=f(u)$ is a differentiable function of *u* and $u=g(x)$ is a differentiable function of *x*, then $y=f\left(g(x)\right)$ is a differentiable function of *x* and

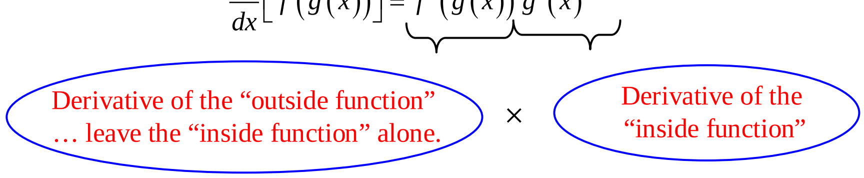
$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

or equivalently,

$$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)g'(x).$$

Composite functions have an "inside function" and an "outside function". Another way to look at this would be

$$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)g'(x)$$



The toughest part (at first) is learning to identify the "inside" and "outside" functions.

Example 2: Each of the following examples can be done without using the chain rule. First state how to find the derivative without using the chain rule, and then use the chain rule to differentiate. State the "inside" and "outside" parts.

a) $f(x)=\dfrac{2}{3x+1}$

b) $g(x)=\left(x^2+2\right)^3$

c) $h(x)=\sin(2x)$

Combining ALL the rules: Power, Product, Quotient, and Chain

Example 3: Find $k'(x)$, if $k(x)=\left(x^2+1\right)\sqrt{2x-3}$.

Example 4: Find $\dfrac{dg}{dt}$, if $g=\left(\dfrac{t-2}{2t+1}\right)^9$

Example 5: For each of the following, use the fact that $g(5)=-3$, $g'(5)=6$, $h(5)=3$, and $h'(5)=-2$ to find $f'(5)$, if possible. If it is not possible, state what additional information is required.

a) $f(x)=g(x)h(x)$

b) $f(x)=g(h(x))$

c) $f(x)=\dfrac{g(x)}{h(x)}$

d) $f(x)=\left[g(x)\right]^3$ … ♬: Your book refers to $\left[g(x)\right]^3$ as $g^3(x)$

3.7 IMPLICIT DIFFERENTIATION

Implicit and Explicit Functions

Equations that are solved for *y* are called explicit functions, whereas equations that are not solved for *y* are called implicit.

For instance, the equation $x + 2y - 3 = 0$, implies that *y* is a function of *x*, even though it is not written in the form $y = -\frac{1}{2}x + \frac{3}{2}$. Up to this point in this class we have been using functions of *x* expressed in the form $y = f(x)$ such as

$$y = \frac{x+1}{x+2} \qquad \text{or} \qquad y = \sin x.$$

If we have an equation that involves both *x* and *y* in which *y* has not been solved for *x*, then we say the equation defines *y* as an *implicit* function of *x*. In this case, we may (or may not) be able to solve for *y* in terms of *x* to obtain an explicit function (or possibly several functions).

First a few skills that you will need …

Example 1: Find the derivative of the following expressions:

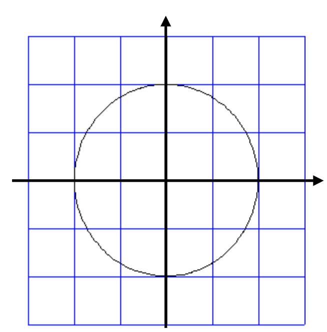
a) Find $\frac{dy}{dx}$: x^3

b) Find $\frac{dy}{dx}$: y^3

c) Find $\frac{dy}{dx}$: xy

d) Find $\frac{dy}{dx}$: $x^3 + y^2 - 3xy$

Example 2: Solve the equation $x^2 + y^2 = 4$ to obtain *y* written as an explicit function of *x*. The graph of this equation is a circle. What is the graph of each explicit function?



Example 3: Find $\frac{dy}{dx}$ for the circle $x^2 + y^2 = 4$:

a) by differentiating the explicit functions of *x*

b) by differentiating implicitly

c) Find where the derivative is positive, where it is negative, where it is zero, and where it is undefined.

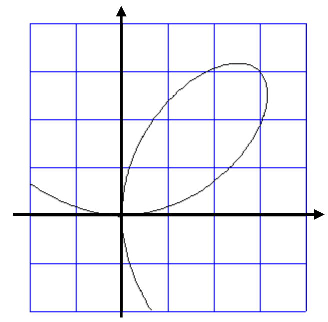
If we have *y* written as an explicit function of *x*, $y = f(x)$, then we know how to compute the derivative $\frac{dy}{dx}$. For an equation which defines *y* as an implicit function of *x*, we can compute the derivative $\frac{dy}{dx}$ without solving for *y* in terms of *x* with the following procedure. **The key to this entire procedure is to remember that even though you did not (or cannot) write *y* as a function of *x*, *y* is implicitly defined as a function of *x*.**

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation **with respect to *x*.** Remember, *y* is a function of *x* (use the Chain Rule)
2. Collect all $\frac{dy}{dx}$ terms on the left side of the equation and move all other terms to the right side of the equation.
3. Factor $\frac{dy}{dx}$ out of the left side of the equation.
4. Solve for $\frac{dy}{dx}$. (It is okay to have both *x*'s and *y*'s in your answer)

To find $\frac{dy}{dx}$ at a given point, plug both the *x* and *y* value into the equation you obtained in step 4.

Example 4: Given the curve $x^3 + y^3 = 6xy$ (shown to the right).



a) Find $\frac{dy}{dx}$.

b) Find the equation of the tangent line and normal (perpendicular) line to the graph at the point $\left(\frac{4}{3}, \frac{8}{3}\right)$.

Example 5: Find $\frac{dy}{dx}$ at (0, 0) of the function $\tan(x + y) = x$.

3.8 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

Notecards from Section 3.8: Derivatives of Inverse Trig Functions, Derivative of an Inverse Function at a Point (p, q)

Derivatives of Inverse Trigonometric Functions

One of the things in the AP Calculus AB Course Description is "Use of implicit differentiation to find the derivative of an inverse function". With that said, let's find the derivative of the inverse sine function using implicit differentiation.

Example 1: Suppose $y = \sin^{-1} x$. Find $\frac{dy}{dx}$ using implicit differentiation.

Derivatives of Inverse Trigonometric Functions where u is a function of x.

1. $\frac{d}{dx}\left[\sin^{-1}(u)\right] = \frac{u'}{\sqrt{1-u^2}}$
2. $\frac{d}{dx}\left[\cos^{-1}(u)\right] = \frac{-u'}{\sqrt{1-u^2}}$
3. $\frac{d}{dx}\left[\tan^{-1}(u)\right] = \frac{u'}{1+u^2}$
4. $\frac{d}{dx}\left[\cot^{-1}(u)\right] = \frac{-u'}{1+u^2}$
5. $\frac{d}{dx}\left[\sec^{-1}(u)\right] = \frac{u'}{|u|\sqrt{u^2-1}}$
6. $\frac{d}{dx}\left[\csc^{-1}(u)\right] = \frac{-u'}{|u|\sqrt{u^2-1}}$

* Domains are restricted to make them functions

* $\sin^{-1}(x)$ and $\arcsin(x)$ are the same thing

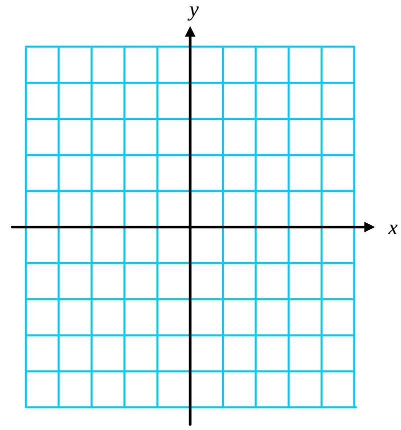
Again … you should commit these to memory as quickly as possible!

Example 2: Find the derivative of $f(t) = \sin^{-1}\left(t^2\right)$

Example 3: Find the derivative of $y = \tan^{-1}\left(\sqrt{x-1}\right)$

Derivatives of Other Inverse Functions

Example 4: Graph the line $y = 4x + 1$



a) What is the slope of the line?

b) Graph the inverse function.

c) What is the slope of the inverse function?

d) If (2, 9) is on the original line, what point does it correspond to on the inverse function?

The slope of the line at (2, 9) on the original function is the _______________ of the slope of the inverse. The difference is that the slope of the inverse is calculated using the point ___________ instead of (2, 9).

New Notation

We have a different notation is used to describe the derivative of f at the point $x = a$. We can write

$$\left.\frac{df}{dx}\right|_a$$

Using the concepts from the last example, we can find the derivative of the inverse function without actually finding the inverse function. We will use the notation above to describe the relationship.

Derivative of the inverse function at a point (p, q) … this implies the point (q, p) is on the original function.

To find the derivative of f^{-1} at the point (p, q) we find the reciprocal of the derivative of f at the point (q, p).

$$\left.\frac{df^{-1}}{dx}\right|_p = \frac{1}{\left.\frac{df}{dx}\right|_q}$$

♬: In other words, if f and g are inverses, then their derivatives at the inverse points are reciprocals.

Example 5: Let $f(x) = x^5 + 2x - 1$. Verify (0, –1) is on the graph. Find $\left(f^{-1}\right)'(-1)$.

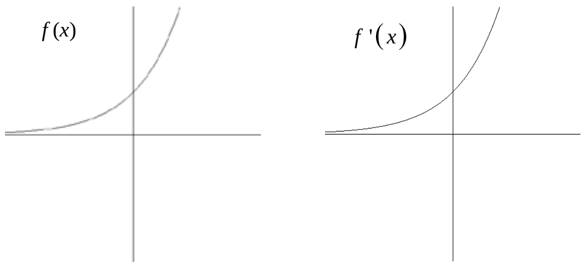
Example 6: Let $f(x) = x^3 + 2x - 1$. Find $\left.\frac{df^{-1}}{dx}\right|_2$.

3.9 DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

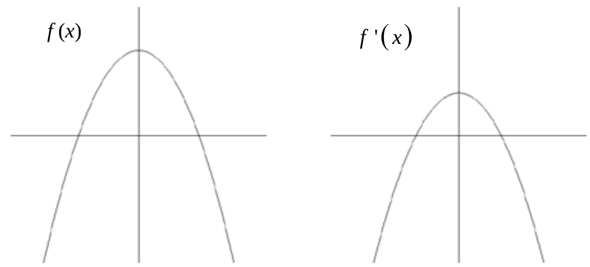
Notecards from Section 3.9: Derivatives of Exponential Functions, Derivatives of Logarithmic Functions, Logarithmic Differentiation

Example 1: Which of the following pairs of graphs could represent the graph of a function AND its derivative?

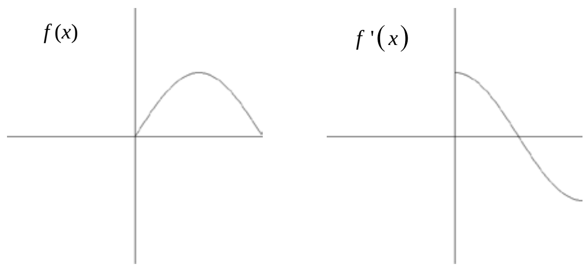
I.



II.



III



A) I only B) II only C) III only D) I and III only E) II and III only

Now for the easiest derivative rule of the year … notice the first pair of graphs (I) above.

Derivative of $f(x) = e^x$

$$\frac{d}{dx}\left[e^x\right] = e^x$$

The proof of this can be done using the definition of a derivative.

The Chain Rule and $f(x) = e^x$

If u is a differentiable function of x, then

$$\frac{d}{dx}\left[e^u\right] = e^u \cdot \frac{du}{dx}.$$

Example 2: Find $\frac{d}{dx}\left[e^{2x-1}\right]$

Example 3: Find $\frac{d}{dx}\left[e^{-3/x}\right]$

The inverse of an exponential function is the natural logarithm function.

Derivative of $f(x) = \ln x$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

Example 4: Prove the derivative rule above using implicit differentiation.

The Chain Rule and $f(x) = \ln x$

If u is a differentiable function of x, then

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx} \qquad \ldots \text{ or } \ldots \qquad \frac{d}{dx}[\ln u] = \frac{u'}{u}$$

Example 5: Let $y = \ln(2x+2)$. Find y'.

Example 6: Let $f(x) = \ln(\tan x)$. Find $f'(x)$

Logarithmic Differentiation

We can use the properties of logarithms to simplify some problems. Here's a quick refresher on those properties.

Definition of a logarithm: $\log_b a = c \iff b^c = a$

Change of Base Formula: $\log_b a = \frac{\log a}{\log b}$ or $\frac{\ln a}{\ln b}$

3 Rules of Logarithms:

1. $\log_b(MN) = \log_b(M) + \log_b(N)$
2. $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$
3. $\log_b(M^k) = k \cdot \log_b(M)$

Example 7: Use the properties of logarithms to rewrite the function, then find the derivative of $y = \log_5 \sqrt{x}$.

Example 8: Use the technique of logarithmic differentiation to find $\frac{dy}{dx}$ for $y=\frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$.

By utilizing the rules of logarithms and implicit differentiation, you can turn an exponential equation into an equation involving logarithms that is usually easier to deal with.

Example 9: Find $\frac{dy}{dx}$ if $y=2^x$.

Example 10: Find $\frac{dy}{dx}$ if $y=3^x$.

The previous two examples lead us to the following result.

Derivative of a^u … *a* is a consant

If $a>0$ and $a\neq 1$ and *u* is a differentiable function of *x*, then

$$\frac{d}{dx}\left[a^u\right]=\ln a\cdot a^u\,\frac{du}{dx}$$

Example 11: Find the derivative of the function $g(x)=e^{5x}+7^x+\ln\left(x^3+4\right)$

4.1 EXTREME VALUES OF FUNCTIONS

Notecards from Section 4.1: Where to Find Extrema, Optimization

Extrema (plural for extremum) are the maximum or minimum values of functions. We need to distinguish between absolute extrema and relative extrema, and how to locate them. You have used your calculator in the past to calculate a maximum or minimum value. In this course, however, you must use calculus reasons to find maximums and minimums!

Definition of Absolute Extrema … The BIGGEST or smallest y-value in the interval.

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum of *f* on the interval *I*** if $f(c) \le f(x)$ for all x in I.
2. $f(c)$ is the **maximum of *f* on the interval *I*** if $f(c) \ge f(x)$ for all x in I.

The Extreme Value Theorem

If f is continuous on a closed interval [a, b], then f has both a minimum and a maximum on the interval

Example 1: Draw a sketch, and find the absolute extrema of the function $f(x)=\sqrt{4-(x+2)^2}$ on the interval If no maximum or minimum exists, which part of the extreme value theorem hypothesis isn't satisfied?

(a) [–4, 0]

(b) [–2, 0)

(c) (–4, –2)

(d) [1, 2)

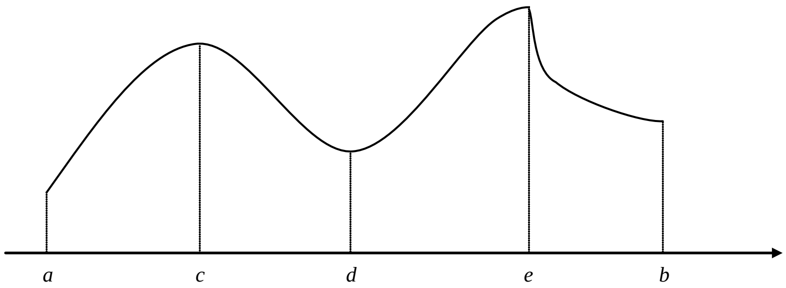
Relative Extrema and Critical Numbers

Definition of Relative Extrema

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum**.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum**.

Basically relative extrema exist when the value of the function is larger (or smaller) than all other function values relatively close to that value.

Example 2: Suppose you were given the function below. Label $f(a)$ through $f(e)$ as absolute or relative extrema.



When given a graph it is fairly simple to identify the extrema. The question to be asked then is how do we find the extrema when we do not have a graph given to us?

Example 3: Except at the endpoints *a* and *b*, what do you notice about the derivative at the relative extrema in the last example?

Definition of a Critical Point

Let f be defined at c. If $f'(c) = 0$ <u>or</u> if f' is undefined at c, then c is a **critical point** of f.

Relative Extrema Occur Only at Critical Points

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

IMPORTANT ♬: Just because the derivative is equal to zero (or undefined) does not mean there is a relative maximum or minimum there. *Relative extrema* can occur ONLY at critical points, and critical points occur ONLY when the derivative is either 0 or undefined, however, it is possible for the derivative to equal zero (or undefined) and there be NO extrema there.

***If you need to convince yourself of this, try $f(x) = x^3$ and $f(x) = \sqrt[3]{x}$ at $x = 0$. …
Is the derivative zero? Undefined? … Is there a maximum or a minimum?

Guidelines for Finding Absolute/Relative Extrema on a Closed Interval

1. Find the critical numbers of f in (a, b). Do this by ____.

2. Evaluate f at each critical number in (a, b).

3. Evaluate f at the endpoints of $[a, b]$. DON'T FORGET THIS PART IF IT THE INTERVAL IS CLOSED!!!!

4. The least of the values from steps 2 and 3 is the absolute minimum, and the greatest of these values is the absolute maximum. If the interval is closed and the endpoints do no result in an absolute max or min, a sign chart can be used to determine whether or not the endpoints result in a relative max or min.

Example 4: The critical numbers (or critical points) are _______ values, while the maximums/minimums of the function are _______ values. In other words, if the point (2, 70) is a relative minimum, the minimum of the function is ________, and it occurs at _______ . (This is how you correctly describe extrema.)

Example 5: Find the extrema of $f(x)=3x^4-4x^3$ on the interval [–1, 2]. Use your graphing calculator to investigate first.

Example 6: Find the extrema of $g(x)=2x-3x^{2/3}$ on the interval [–1, 3]. Use your graphing calculator to investigate first.

Optimization … really §4.4

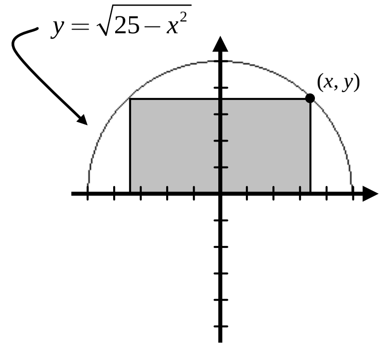
All optimization refers to is finding the absolute maximum or minimum. The usual difference between "optimization" problems and problems we have done so far is that the equation must be written by you.

Remember, ***if you have a closed interval you MUST test the endpoints of the function as well.***

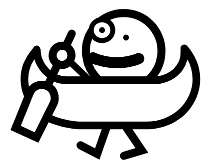
Steps for Solving Applied Optimization Problems

Step 1: Understand the problem. Read it carefully, and ask yourself, "Self, what is the quantity to be maximized or minimized? What are the quantities which it depends on?"
Step 2: Draw a diagram if possible. Sometimes, more than one diagram helps you to determine how all the quantities are related. Identify all quantities from step 1 on your diagram.
Step 3: Assign variables to the quantities from step 1.
Step 4: Determine a function for the quantity to be optimized.
Step 5: Eliminate all but ONE variable and determine the domain of the resulting function. If this is necessary, you will need a second equation that relates the variables in the original equation.
Step 6: Optimize the function. (Find the absolute MAX or MIN)
(a) Calculate the derivative and find the critical numbers
(b) If the domain is a closed interval, compare the function's value of the critical numbers with that of the endpoints.
(c) If the domain is an open interval (or infinite interval), use the *first or second derivative tests* to analyze the behavior of the function. (*This will be explained in the next few sections*)

Example 7: A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$. What length and width should the rectangle have so that its area is a maximum?



Example 8: You are in a rowboat on Lake Perris, 2 miles from a straight shoreline taking your potential in-laws for a boat ride. Six miles down the shoreline from the nearest point on shore is an outhouse. You suddenly feel the need for its use. It is November, so the water is too cold to go in, and besides, your in-laws are already pretty unimpressed with your “yacht”. It wouldn’t help matters to jump over the side and relieve your distended bladder. Also, the shoreline is populated with lots of houses, all owned by people who already have restraining orders against you (apparently you've been out here before!). If you can row at 2 mph and run at 6 mph (you can run faster when you don’t have to keep your knees together), for what point along the shoreline should you aim in order to minimize the amount of time it will take you to get to the outhouse? (…And you thought calculus wasn’t useful!)



4.2 MEAN VALUE THEOREM

Notecards from Section 4.2: Mean Value Theorem; Relationship of f' to Increasing/Decreasing

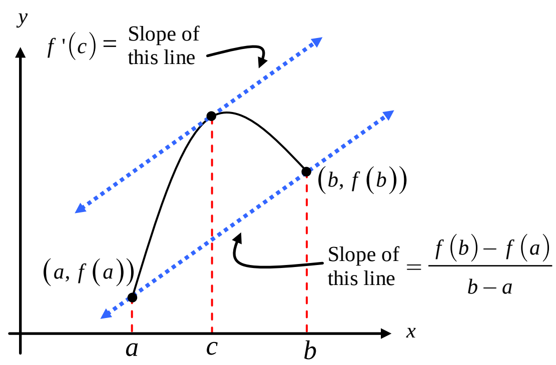
The Mean Value Theorem is considered by some to be the most important theorem in all of calculus. It is used to prove many of the theorems in calculus that we use in this course as well as further studies into calculus.

The Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then *there exists* a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Just like the Intermediate Value Theorem, this is an *existence theorem*. The Mean Value Theorem does not tell you what the value of c is, nor does it tell you have many exist. Again, just like the Intermediate Value Theorem, you must keep in mind that c is an x – value.



Also, the hypothesis of the Mean Value Theorem (MVT) is highly important. If any part of the hypothesis does not hold, the theorem cannot be applied.

Basically, the Mean Value Theorem says, that the average rate of change over the entire interval is equal to the instantaneous rate of change at some point in the interval.

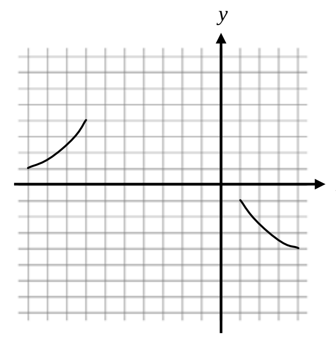
Example 1: A plane begins its takeoff at 2:00 pm on a 2500-mile flight. The plane arrives at its destination at 7:30 pm (ignore time zone changes). Explain why there were at least two times during the flight when the speed of the plane was 400 miles per hour.

Example 2: Apply the Mean Value Theorem to the function on the indicated interval. In each case, make sure the hypothesis is true, then find all values of c in the interval that are guaranteed by the MVT.

a) $f(x) = x(x^2 - x - 2)$ on the interval $[-1, 1]$

b) $f(x) = \dfrac{x+1}{x}$ on the interval $[0.5, 2]$.

Example 3: The figure to the right gives two parts of the graph of a differentiable Function f on [–10, 4]. The derivative f' is also continuous.



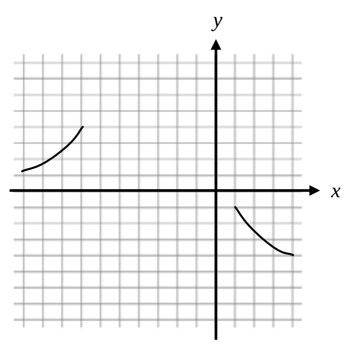
a) Explain why f must have at least one zero in (–7, 1).

b) Explain why there must be at least one point in the interval (–7, 1) whose derivative is $-\frac{5}{8}$.

c) Explain why f' must also have at least one zero in the interval [–10, 4]. What are these zeros called?

d) Make a possible sketch of the function with one zero of f' on the interval [–10, 4].

e) Make a possible sketch below of the function with exactly two zeros of f' on the interval [–10, 4].



While the Mean Value Theorem is used to prove a wide variety of theorems, we will be focusing on the results and/or consequences of the Mean Value Theorem. In this section, we will discuss when a function increases and decreases as well as a brief introduction to antiderivatives.

Increasing versus Decreasing

Why mathematicians feel the need to define everything is a mystery you will probably never figure out unless you become one. Then, for some inexplicable reason, you will find yourself questioning the truthfulness of every argument ever made, reducing every argument to its basic foundational vocabulary, and finally analyzing the very soul and fiber of the definitions. With this in mind, we will now define what it means for a function to be increasing and decreasing. (Obviously, we cannot just say that a function is increasing when all the function values get bigger.)

Definitions of Increasing and Decreasing Functions

A function *f* is **increasing** on an interval if for any two numbers x_1 and x_2 in the interval,

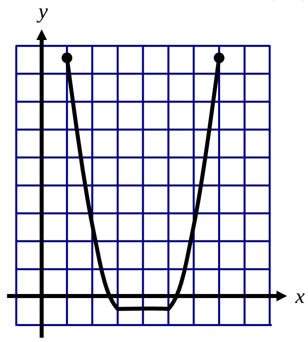
$$x_1 < x_2 \text{ implies } f(x_1) < f(x_2).$$

A function *f* is **decreasing** on an interval if for any two numbers x_1 and x_2 in the interval,

$$x_1 < x_2 \text{ implies } f(x_1) > f(x_2).$$

Example 4: What interval is the function decreasing? increasing? constant?

Example 5: What is the value of the derivative when the function is decreasing? increasing? constant?



Test for Increasing and Decreasing Functions

Let *f* be a function that is continuous on the closed interval [*a*, *b*] and differentiable on the open interval (*a*, *b*).

1. If $f'(x) > 0$ <u>for all</u> *x* in (*a*, *b*), then *f* is increasing on [*a*, *b*].
2. If $f'(x) < 0$ <u>for all</u> *x* in (*a*, *b*), then *f* is decreasing on [*a*, *b*].
3. If $f'(x) = 0$ <u>for all</u> *x* in (*a*, *b*), then *f* is constant on [*a*, *b*].

Guidelines for Finding Intervals on Which a Function is Increasing or Decreasing

Let *f* be continuous on the interval (*a*, *b*). To find the open intervals on which *f* is increasing or decreasing use the following steps:

1. Find the <u>critical points</u> of *f* in the interval (*a*, *b*), and use these numbers to create a sign chart.
2. Determine the sign of $f'(x)$ at ONE test value in each interval. (label your sign chart)
3. Use the <u>signs of the derivative</u> to determine whether or not the function is increasing or decreasing.
4. Your sign chart is not enough to justify your response. Your response should be worded …
   "The function is increasing (or decreasing) on the interval (*c*, *d*) since $f'(x) > 0$ (or $f'(x) < 0$) "

Example 6: Find the intervals on which $f(x) = x^3 - 12x - 5$ is increasing or decreasing.

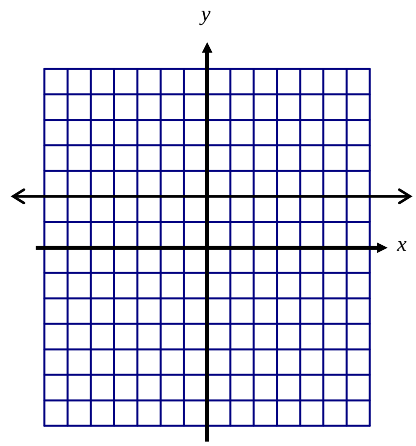
Example 7: Find the intervals on which $f(x)=\left(x^2-9\right)^{2/3}$ is increasing or decreasing.

Antiderivatives

Example 8: Suppose you were told that $f'(x)=2x-1$. What could $f(x)$ possibly be? Is there more than one answer?

Finding the function from the derivative is a process called ***antidifferentiation***, or finding the antiderivative.

Example 9: Suppose the graph of $f'(x)$ is given to the right.
Draw at least three possible functions for $f(x)$.
(*Hint*: If the derivative is given, then the *y*-values of the derivative are the slopes of the original function.)



The three functions you drew should only differ by a constant. If you let C represent this constant, then you can represent the *family* of all antiderivatives of $f'(x)$ to be $f(x)=2x+C$.

Example 10: If you were told that $f(3)=-2$, what would the value of C be?

IMPORTANT ♬: If a function has one antiderivative it has many antiderivatives that all differ by a constant. Unless you know something about the original function, you cannot determine the exact value of that constant, but it must be in your answer!

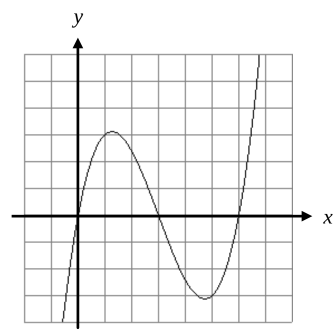
4.3 CONNECTING f' AND f'' WITH THE GRAPH OF f

Notecards from Section 4.3: 1st Derivative Test for Extrema; Relationship of f'' to Concavity; 2nd Derivative Test for Extrema; Relationship of f'' to Inflection Points

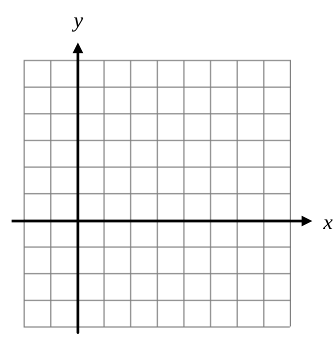
First Derivative Test for Extrema

We have already determined that relative extrema occur at critical points. The behavior of the first derivative before and after those critical points will help determine whether or not the function has a relative maximum or minimum (or neither) at these critical points.

Example: Given the graph of f below, label the relative extrema.



Example: Sketch f' as accurately as you can below. Label the x-values of the extrema from f on f'.



Check #1 and #2 in the box below with the graph you just sketched.

The First Derivative Test

Let f be a continuous function, and let c be a critical point.

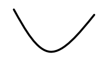
1. If f' changes sign from positive to negative at c, then f has a local maximum value at c.
2. If f' changes sign from negative to positive at c, then f has a local minimum value at c.
3. If f' DOES NOT change signs, then there is no local extreme value at c.

Important ♬: If you are asked to find the absolute maximum (or just a maximum) of a function on a closed interval, you MUST test the endpoints also, and it may be just as simple to plug in the endpoints and the critical points.

Example: Find where the function $h(x) = x\sqrt{4-x^2}$ is increasing and decreasing, then use the first derivative test to determine any local extrema.

Example: Find where the function $g(x) = x^2e^x$ is increasing and decreasing, then find any local extrema and absolute extrema.

Concavity

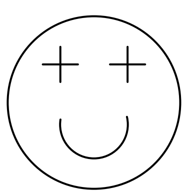
Concavity deals with how a graph is curved. A graph that is concave up looks like , while a graph that is concave down looks like . We can use the SECOND derivative to determine the concavity of a function.

> *Definition of Concavity*
>
> Let $y = f(x)$ be a differentiable function on an interval I. The graph of $f(x)$ is **concave up** on I if f' is increasing on I, and **concave down** on I if f' is decreasing on I.

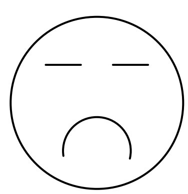
If the first derivative is increasing, then the second derivative must be _________________. If the first derivative is decreasing, then the second derivative must be _________________. Thus instead of using the definition of concavity to determine whether the function is concave up or down, we can use the following test.

> *Concavity TEST*
>
> The graph of a twice-differentiable function $y = f(x)$ is **concave up** on any interval where $y'' > 0$, and **concave down** on any interval where $y'' < 0$.

The Concavity Test can be summed up by the following pictures … While this is a humorous (and hopefully helpful) way to remember concavity, please understand that this is NEVER to be used as a justification on ANY test!

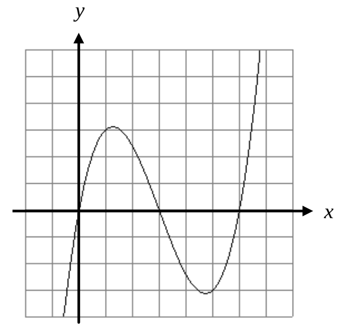


f'' positive $\Rightarrow$ Concave UP

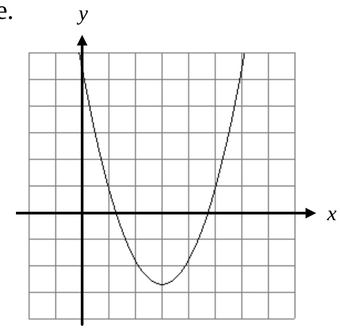


f'' negative $\Rightarrow$ Concave DOWN

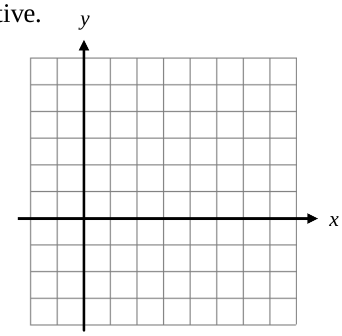
Example: Using the same graph as our previous example, indicate which portions of the graph are concave up, and which portions are concave down. Label the point where the graph changes concavity.



Example: The graph of f' is graphed below. Label the *x*-value where the graph changes concavity. Verify this with the definition of concavity as it relates to the first derivative.



Example: Sketch the graph of f'' on the graph below. Label the *x*-value where the graph changes concavity. Verify this with the definition of concavity as it relates to the second derivative.



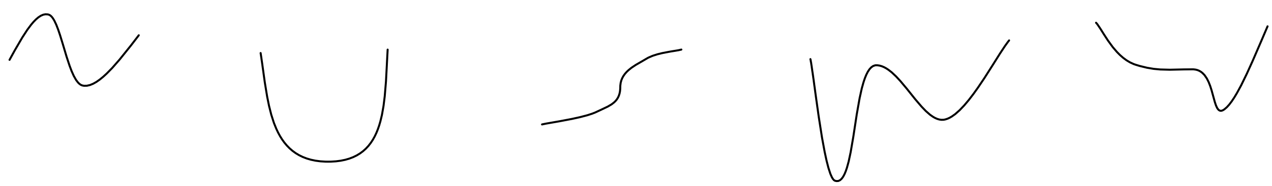
Example: Find the intervals where the function $g(x) = -2x^3 + 6x^2 - 3$ is concave up and concave down.

Example: Determine the concavity of $h(x) = x\sqrt{4-x^2}$.

Points of Inflection

> *Definition*
>
> A point where the graph of a function has a tangent line (even if it's a vertical tangent line) AND where the concavity changes is a **point of inflection**.

Example: Using each picture, estimate each point of inflection, if any, and sketch the tangent line at that point.



> Since points of inflection occur when the graph changes ________________, and a graph changes concavity when the ____________________________________ changes from positive to negative (or vice-versa), then if we wanted to find the points of inflection of a graph, we only need to focus on when the second derivative equals 0 (or does not exist)
>
> ***IMPORTANT*** ♬: Just because the second derivative equals zero (or does not exist) you are NOT guaranteed that the function has a point of inflection. The second derivative MUST change signs (meaning concavity changed) in order for a point of inflection to exist!

Example: Find the points of inflection of $g(x) = -2x^3 + 6x^2 - 3$.

Example: Find the points of inflection of $h(x) = x\sqrt{4-x^2}$.

Second Derivative Test for Extrema

Example: Go back to the original function on page 4 - 9. First look at the point where the function had a relative maximum. Was the graph concave up or down at that point? Secondly, look at the point where the function had a minimum. Was the graph concave up or down at that point?

As long as the function is twice-differentiable (meaning the first derivative is a smooth curve), then we can actually determine whether or not a critical point is a relative maximum or minimum WITHOUT testing values to the right and left of the point. We can use the Second Derivative Test.

Second Derivative Test for Local Extrema

If $f'(c)=0$ (which makes $x = c$ a critical point) AND $\boldsymbol{f''(c)<0}$, then *f* has a local MAXIMUM at $x = c$.
If $f'(c)=0$ (which makes $x = c$ a critical point) AND $\boldsymbol{f''(c)>0}$, then *f* has a local MINIMUM at $x = c$.

Important ♬: If the second derivative is equal to zero (or undefined) then the Second Derivative Test is INCONCLUSIVE.

Remember the happy (and sad) faces? If a critical point happens to occur in an interval where the graph of the function is CONCAVE UP, then that critical point is a relative MINIMUM. If a critical point happens to occur in an interval where the graph of the function is CONCAVE DOWN, then that critical point is a relative MAXIMUM.

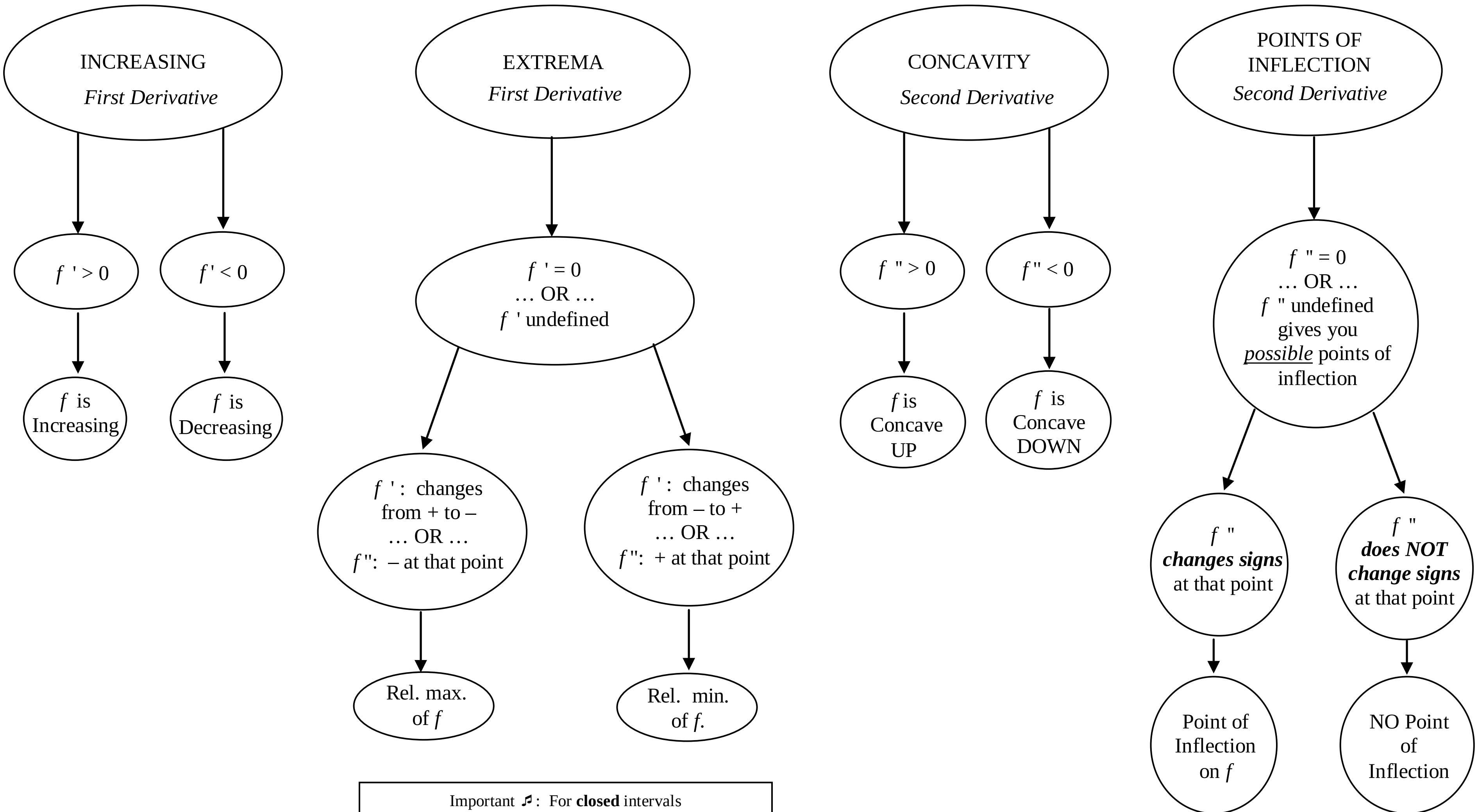
Example: Use the Second Derivative Test to identify any relative extrema for the function $g(x)=-x^4+4x^3-4x+1$.

NOTES FOR OPTIMIZATION PROBLEMS:

Whenever you are required to Maximize or Minimize a function, you MUST justify whether or not your answer is actually a maximum or a minimum. You may use the FIRST DERIVATIVE TEST (testing points to the left and right of the critical points in the first derivative to see if the sign of the first derivative changes from positive to negative or vice-versa), or the SECOND DERIVATIVE TEST (plugging in the critical points to the second derivative to see if the critical points occur when the original function was concave up or down).

ALWAYS REMEMBER that both of these tests are checking for relative extrema. If you have a CLOSED interval, you must check the endpoints to make sure the absolute maximum or minimum values do not happen to occur there. If you have a closed interval, it is best just to check ALL critical points and endpoints.

A QUICK SUMMARY OF SECTIONS 4.2 AND 4.3



Important ♬: For **closed** intervals
check the endpoints to find absolute extrema