Calculus
Chapter 3.1 – 3.5 Review Sheet Solutions

1. f is NOT differentiable at
    $x = -2$, $x = 1$, and $x = 2$ because of a "point place" ,
    $x = 3$ because of discontinuity …
    and one could argue at $x = -1/2$ because of a vertical tangent line,

    …although to be fair, most of the time a vertical tangent line is involved, there will be a comment about the vertical tangent line in the problem.

2. To find marginal profit, find the derivative of P … $P'(x) = 12x^2 - 7$

    The marginal profit of the 12th item (currently producing 11 items) is $P'(11) = 12(11)^2 - 7 = 1445$

3. For the derivative at $x = 2$ to exist the function must be continuous at 2 and the derivative from the left and right sides must be the same.

4. First find the derivative of the function: $f'(x) = \begin{cases} 4ax & x > 1 \\ -3 & x < 1 \end{cases}$.

In order for the derivative to exist at $x = 1$, the derivative from the left and the derivative from the right must be equal. So, $4a(1) = -3$, meaning $a = \dfrac{-3}{4}$.

In order for f to be continuous at $x = 1$, $\lim_{x\to 1} f(x) = f(1)$. In order for this limit to exist, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$.
So, evaluating the left and right hand limits and substituting what we now know a equals we get

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^+} f(x) = f(1) \\ -3(1) + 4 &= 2a(1)^2 + b \\ 1 &= 2\left(\tfrac{-3}{4}\right) + b \\ \tfrac{5}{2} &= b \end{aligned}$$

Therefore, to be continuous and differentiable, $a = \dfrac{-3}{4}$ and $b = \dfrac{5}{2}$.

5. a) Average velocity is $\dfrac{\text{total distance}}{\text{total time}} = \dfrac{x(3)-x(0)}{3-0} = \dfrac{-3-12}{3} = -5$ ft/s

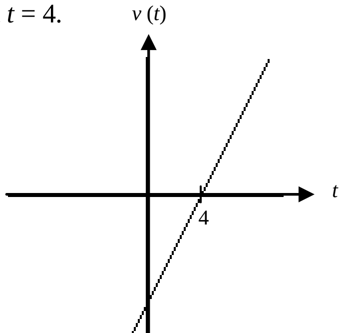
b) Velocity is the derivative of position: $v(t) = 2t - 8$ … $v(4) = 0$ ft/s

c) Since velocity is 0 at t = 4, the object is stopped at t = 4.

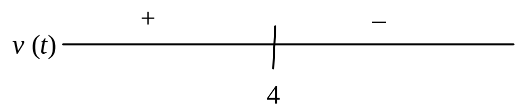
d) Acceleration is the derivative of velocity: $a(t) = 2 \, ft/s^2$. … Thus, acceleration NEVER equals 0.

e) The object changes direction when the velocity changes sign. There are two ways to look at this ….

Graphically: Notice at t = 4, the graph of the velocity function changes from negative to positive, so the object changes direction at t = 4.



Numerically: Use a sign chart … plug in any number to the left and right of t = 4 into the velocity function. Since velocity changes from + to – at t = 4, the object changes direction.



f) The object slows down when the velocity and acceleration have opposite signs. Since the acceleration is always positive, then the object is slowing down when velocity is negative. Thus, when $t < 4$.

g) The object is moving left when velocity is negative. Again, when $t < 4$.

6. The equation of a tangent line requires a point and a slope.

To get the point, plug $x = \frac{\pi}{2}$ into the function. So, the point is $\left(\frac{\pi}{2}, 0\right)$.

To find the slope, take a derivative. Using the PRODUCT RULE (let u = 2sin x and v = cos x) we have

$y' = 2\sin x(-\sin x) + \cos x(2\cos x) = -2\sin^2 x + 2\cos^2 x$.

When $x = \frac{\pi}{2}$, $y' = -2 - 0 = -2$.

So, the equation of the tangent line is $y - 0 = -2\left(x - \frac{\pi}{2}\right)$ or $y = -2x + \pi$. … Try graphing both the function and the tangent line on your calculator as a way to check.

7. The first limit is the limit definition of the derivative of cos x at $x = \frac{\pi}{2}$.
Since the derivative of cos x is –sin x, then the answer is $-\sin\left(\frac{\pi}{2}\right) = -1$.

The second limit is the limit definition of the derivative of $\sqrt{x}$ at x = 4.
Since the derivative of $\sqrt{x}$ is $\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$, then the answer is $\frac{1}{2\sqrt{4}} = \frac{1}{4}$.

8. Using the alternative definition we get

$$\begin{aligned} f'(2) &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} \\ &= \lim_{x\to 2}\frac{\left(\frac{3}{x}-\frac{3}{2}\right)}{x-2} \\ &= \lim_{x\to 2}\frac{\left(\frac{6-3x}{2x}\right)}{x-2} \\ &= \lim_{x\to 2}\frac{3(2-x)}{2x}\cdot\frac{1}{x-2} \\ &= \lim_{x\to 2}\frac{-3}{2x} \\ &= \frac{-3}{4} \end{aligned}$$

You should be able to <u>check</u> this using the power rule … since $f(x)=\frac{3}{x}=3x^{-1}$, then $f'(x)=-3x^{-2}=\frac{-3}{x^2}$.

Therefore, $f'(2)=\frac{-3}{4}$.

9. Using the alternative definition we get

$$\begin{aligned} f'(1) &= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} \\ &= \lim_{x\to 1}\frac{(3x^2+5x)-(8)}{x-1} \\ &= \lim_{x\to 1}\frac{3x^2+5x-8}{x-1} \\ &= \lim_{x\to 1}\frac{(3x+8)(x-1)}{x-1} \\ &= \lim_{x\to 1}(3x+8) \\ &= 11 \end{aligned}$$

You should be able to <u>check</u> this using the power rule … since $f(x)=3x^2+5x$, then $f'(x)=6x+5$. Therefore, $f'(1)=6(1)+5=11$.

10. $f'(3)$ is the derivative at $x = 3$ … or the rate of change in \$/min at 3 minutes. To approximate this value, use the slope of a secant line (any secant line that includes $x = 3$ is technically ok)

$$\frac{f(4)-f(3)}{4-3}=\frac{11-9}{1}=2$$

$$\frac{f(3)-f(2)}{3-2}=\frac{9-6}{1}=3$$

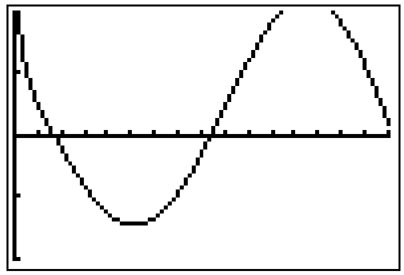
$$\frac{f(4)-f(2)}{4-2}=\frac{11-6}{2}=\frac{5}{2}$$

11. a) Velocity is the derivative of position (use product rule) … $v(x) = s'(x) = \sqrt{x}(-\sin x) + \cos x\left(\frac{1}{2}x^{-1/2}\right)$

b) The acceleration is the derivative of the velocity function (or second derivative of position) … use the product rule for each part … $a(x) = s''(x)$

$$= \left[\sqrt{x}(-\cos x) + (-\sin x)\left(\tfrac{1}{2}x^{-1/2}\right)\right] + \left[\cos x\left(-\tfrac{1}{4}x^{-3/2}\right) + \left(\tfrac{1}{2}x^{-1/2}\right)(-\sin x)\right]$$

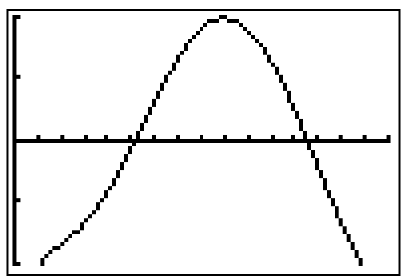
c) The object is stopped when velocity equal 0 … Setting your window for x to be $0 \le x < 2\pi$ and $-2 \le y \le 2$ you get the following image



using a calculator to solve for the zeros, you find the object is stopped at $x \approx .65327119$, and $x \approx 3.29231$.

d) The particle changes direction when the velocity changes sign … once again, using the graph, the velocity changes sign when $x \approx .653$ and $x \approx 3.292$.

e) The particle is speeding up when the velocity and acceleration have the same sign … using your calculator, graph the acceleration function and find the zeros so that you can determine when the acceleration changes sign (using the same window as before, you get the image below)



The zeros of the acceleration function are $x \approx 2.0090972$, and $x \approx 4.9112508$. The object is speeding up when velocity and acceleration are both positive … on the interval (3.292, 4.911) … and when the velocity and acceleration are both negative … on the interval (.653, 2.009).

f) Zeros of $s(x)$ would be where the position function crosses the x-axis … $x = 0$, $\pi/2$, and $3\pi/2$

g) Zeros of $v(x)$ are $x \approx .65327119$, and $x \approx 3.29231$.

h) Zeros of $a(x)$ are $x \approx 2.0090972$, and $x \approx 4.9112508$.

12. If $f(x)$ has a derivative at $x = 2$, then it must also be continuous at $x = 2$… so

a … MUST BE TRUE (part of the definition of continuity),

b … MUST BE TRUE (because the $f'(2)$ is notation for the derivative at $x = 2$),

c … is not necessarily true because we have been given nothing about the second derivative,

d … MUST BE TRUE (because if a function has a derivative at a point, then it must be continuous at that point),

e … MUST BE TRUE (because that is the alternative definition of the derivative at $x = 2$), and

f … MUST BE TRUE (because that is the original definition of the derivative at $x = 2$).