Calculus
Chapter 2 Review Solutions

I have done my best to make sure all the solutions are correct. Inevitably, there seem to be typos. If you do not agree/understand a solution, email me or find time to ask me about them BEFORE the exam.

1. Use direct substitution. $\lim_{x\to 5/2}\lfloor x\rfloor = \lfloor \frac{5}{2}\rfloor = 2$

2. The numerator has a higher degree than the denominator, so $\lim_{x\to\infty}\frac{x^2+5x-3}{3x+2}$ does not exist (DNE).

3. The numerator and denominator have the same degree, so $\lim_{x\to\infty}\frac{x^2+5x-3}{3x^2+2}=\frac{1}{3}$

4. The numerator has a lower degree than the denominator, so $\lim_{x\to\infty}\frac{x^2+5x-3}{3x^3+2}=0$

5. Multiply the denominator by $\frac{2x}{2x}$ … $\lim_{x\to 0}\frac{x}{\sin(2x)}\cdot\frac{}{\frac{2x}{2x}}=\lim_{x\to 0}\frac{\cancel{x}}{\frac{\sin(2x)}{2x}\cdot 2\cancel{x}}=\lim_{x\to 0}\frac{1}{\frac{\sin(2x)}{2x}\cdot 2}=\frac{1}{1\cdot 2}=\frac{1}{2}$

6. Take out the $\frac{1}{2}$ to get $\lim_{x\to\infty}\frac{\sin x}{2x}=\frac{1}{2}\lim_{x\to\infty}\frac{\sin x}{x}=\frac{1}{2}\cdot 0=0$ … or realize that as $x\to\infty$, the denominator gets really large while the numerator stays between 1 and –1.

7. Rewrite tan (5*x*) in terms of sine and cosine and multiply by 1/sin(3*x*) instead of dividing …

$\lim_{x\to 0}\frac{\tan(5x)}{\sin(3x)}=\lim_{x\to 0}\frac{\sin(5x)}{\cos(5x)}\cdot\frac{1}{\sin(3x)}$ … Multiply the numerator by $\frac{5x}{5x}$ … and the denominator by $\frac{3x}{3x}$

$$\lim_{x\to 0}\frac{\sin(5x)\cdot\frac{5x}{5x}}{\cos(5x)}\cdot\frac{1}{\sin(3x)\cdot\frac{3x}{3x}}=\lim_{x\to 0}\frac{\frac{\sin(5x)}{5x}\cdot 5\cancel{x}}{\cos(5x)}\cdot\frac{1}{\frac{\sin(3x)}{3x}\cdot 3\cancel{x}}$$

Once the *x* is cancelled above, you can evaluate the limit … $\lim_{x\to 0}\frac{\frac{\sin(5x)}{5x}\cdot 5}{\cos(5x)}\cdot\frac{1}{\frac{\sin(3x)}{3x}\cdot 3}=\frac{1\cdot 5}{1}\cdot\frac{1}{1\cdot 3}=\frac{5}{3}$

8. The numerator grows faster than the denominator, so $\lim_{x\to\infty}\frac{4x^2+5x}{x-3}$ does not exist (DNE).

9. The numerator and denominator grow at the same rate, so $\lim_{x\to\infty}\frac{5x-7x^2}{4x^2+1}=\frac{-7}{4}$

10. A graph of this (make a table if necessary) would show that the $\lim_{x\to -3^+}\frac{|x+3|}{x+3}=1$ and $\lim_{x\to -3^-}\frac{|x+3|}{x+3}=-1$ meaning $\lim_{x\to -3}\frac{|x+3|}{x+3}$ does not exist (DNE).

11.

x	1000	10000	100000	1000000	10000000	100000000
$\left(1+\frac{1}{x}\right)^x$	2.716923932	2.718145927	2.718268237	2.718280469	2.718281694	2.718281786

Recognize the number? It's *e* … <u>Add this to your notecards</u> under "Limits you should know". … yeah … those things you have to hand in before your test! ☺

12. Since you're approaching 2 from both sides, pick two numbers close to 2 ***<u>on both sides</u>***.

x	$\frac{x+3}{x-2}$
1.99	–499
1.999	–4999
2.001	5001
2.01	501

Based on the table, $\lim_{x\to 2^-}\frac{x+3}{x-2}$ DNE, and as $x \to 2^-$, $\frac{x+3}{x-2} \to -\infty$.

Also, as $x \to 2^+$, $\frac{x+3}{x-2} \to \infty$.

For questions 13 and 14, find ALL asymptotes (vertical, horizontal, and oblique) and justify your response.

13. This function has a vertical asymptote at $x = 0$ because as $x \to 0^+$, $\ln x \to -\infty$. (A parent function you should know)

14. Since $f(x) = \frac{\cancel{(x+2)}(x-3)}{\cancel{(x+2)}(x-1)} = \frac{x-3}{x-1}$ there is NOT a vertical asymptote at $x = –2$ (there's a hole), but there IS a vertical asymptote at $x = 1$, since $\lim_{x\to 1}\frac{x-3}{x-1}$ DNE because as $x \to 1^-$, $\frac{x-3}{x-1} \to \infty$ and as $x \to 1^+$, $\frac{x-3}{x-1} \to -\infty$ Also, since $\lim_{x\to\infty}\frac{(x+2)(x-3)}{(x+2)(x-1)} = 1$, there is a horizontal asymptote at $y = 1$.

15. Since $h(x) = \frac{(x-1)\cancel{(x+3)}}{\cancel{(x+3)}(x-2)} = \frac{x-1}{x-2}$, there is a vertical asymptote at $x = 2$, but there is a hole when $x = –3$. Therefore, $\lim_{x\to c} h(x)$ exists for all real numbers except for $x = 2$.

16.

a) Since $g(x) = \frac{x^2+5x+6}{x^2+3x+2} = \frac{\cancel{(x+2)}(x+3)}{(x+1)\cancel{(x+2)}} = \frac{x+3}{x+1}$, the domain is $x \neq -1$ and $x \neq -2$

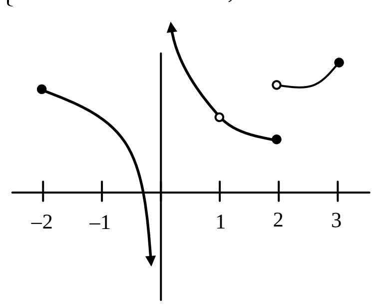
b) Since there is a removable discontinuity (a hole) at $x = –2$, the limit as x approaches –2 exists, but since there is a vertical asymptote at $x = –1$, the limit does not exist as x approaches –1.

c) Since $\lim_{x\to\infty}\frac{x^2+5x+6}{x^2+3x+2} = 1$, there is a horizontal asymptote at $y = 1$.

d) There is a vertical asymptote at $x = –1$, since $\lim_{x\to -1^+}\frac{x^2+5x+6}{x^2+3x+2} = \lim_{x\to -1^+}\frac{x+3}{x+1}$ which does not exist (DNE) and as $x \to -1^+$, $\frac{x+3}{x+1} \to \infty$.

e) The only value of x less than –1 where $g(x)$ is not continuous is $x = –2$.
Since there is a removable discontinuity at $x = –2$, we just need to define the value of $g(–2)$ to fill the hole. Using the simplified version of $g(x)$, we see that $g(-2) = \frac{-2+3}{-2+1} = -1$. Thus we write $g(x) = \begin{cases} \frac{x^2+5x+6}{x^2+3x+2} & ; x \neq -1 \text{ and } x \neq -2 \\ -1 & ; x = -2 \end{cases}$

17. Using the function below, over what intervals does $\lim_{x\to c} f(x)$ exist?



The limit fails to exist at $x = 0$ and at $x = 2$ (both are non-removable discontinuities) .
The limit does exist at $x = 1$, even thought there is a hole (removable discontinuity).

So, the intervals where $\lim_{x\to c} f(x)$ exists are $(-2,0)\cup(0,2)\cup(2,3)$.

♫: The $\lim_{x\to -2^+} f(x)$ also exists, as does $\lim_{x\to 3^-} f(x)$.

18. Let $y = x^3 - 4x$.

a) $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left[(x+h)^3-4(x+h)\right]-\left(x^3-4x\right)}{h}$$
$$= \lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3\cancel{-4x}-4h\cancel{-x^3}+\cancel{4x}}{h}$$
$$= \lim_{h\to 0}\frac{\cancel{h}\left(3x^2+3xh+h^2-4\right)}{\cancel{h}}$$
$$= \lim_{h\to 0}\left(3x^2+3xh+h^2-4\right)$$
$$= 3x^2+0+0-4$$
$$= 3x^2-4$$

b) Slope at $x = -1$: $3(-1)^2 - 4 = -1$

c) When $x = -1$, $y = 3$ and the slope is -1: $y-3=-1(x+1)$

d) The slope of the normal line would be +1: $y-3=1(x+1)$

19.
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$$\lim_{h\to 0}\frac{f(4+h)-f(4)}{h} = \lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\cdot\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$
$$= \lim_{h\to 0}\frac{(4+h)-4}{h\left(\sqrt{4+h}+2\right)}$$
$$= \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}\left(\sqrt{4+h}+2\right)}$$
$$= \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}$$
$$= \frac{1}{4}$$

For questions 20 - 22, find the value of the parameter(s) that would make the function continuous. Justify your response using the definition of continuity.

20. $$j(x) = \begin{cases} ax^2 & ; x < 1 \\ 4x-2 & ; x \geq 1 \end{cases}$$

In order to be continuous, the only problem occurs when $x = 1$. To be continuous at $x = 1$, $\lim_{x\to 1} f(x) = f(1)$. In order for $\lim_{x\to 1} f(x)$ to exist, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$.

$$a(1)^2 = 4(1) - 2$$
$$a = 2$$

Thus, if $a = 2$, $\lim_{x\to 1} f(x) = 2 = f(2)$, and $f(x)$ is continuous.

21. In order to be continuous, the only problem occurs when $x = 0$. To be continuous at $x = 0$, $\lim_{x\to 0} k(x) = k(0)$.

$$\lim_{x\to 0}\frac{\sin 3x}{x} = a$$
$$\lim_{x\to 0}\frac{\sin 3x}{x}\cdot\frac{3}{3} = a$$
$$3\lim_{x\to 0}\frac{\sin 3x}{3x} = a$$
$$3\cdot 1 = a$$
$$3 = a$$

Thus, if $a = 3$, $\lim_{x\to 0} k(x) = 3 = k(0)$, and $k(x)$ is continuous.

22. Let $y = \dfrac{x^2+5x-3}{x-2}$.

a) End Behavior Model: $\dfrac{x^2}{x} = x$

b) End Behavior: as $x\to\infty$, $y\to\infty$, and as $x\to-\infty$, $y\to-\infty$

c) There is a vertical asymptote at $x = 2$. Since the $\lim_{x\to\infty} y$ does not exist (DNE), there are no horizontal asymptotes. However, using long division you get $\dfrac{x^2+5x-3}{x-2} = x+7+\dfrac{11}{x-2}$. Therefore, there is a slanted (oblique) asymptote at $y = x + 7$.

23. When $x_0 = 0$, $\lim_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$

$$= \lim_{h\to 0}\frac{h^{2/5}-0}{h}$$
$$= \lim_{h\to 0}\frac{1}{h^{3/5}}$$

This limit does not exist, since as $h\to 0^+$, $\dfrac{1}{h^{3/5}}\to\infty$, and as $h\to 0^-$, $\dfrac{1}{h^{3/5}}\to-\infty$. Thus according the statement made in the problem, there is a vertical tangent line at $x = 0$. … Try looking at a graph to convince yourself of this. (You may want to zoom in close to the origin.)

24. In order for $k(x)$ to be continuous at $x = 9$, we need $\lim_{x\to 9} k(x) = k(9)$. Since $k(9)$ is currently not defined, we just need to evaluate the limit and then assign the same value to $k(9)$. In order to evaluate the limit, we must rationalize the numerator because if you just plug in $x = 9$, you get $\frac{0}{0}$.

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x\to 9}\frac{\cancel{x-9}}{\cancel{(x-9)}\left(\sqrt{x}+3\right)} = \lim_{x\to 9}\frac{1}{\sqrt{x}+3} = \frac{1}{6}$$

Since $k(9) = \frac{1}{6}$, we now have the extended (and continuous) version of $k(x) = \begin{cases}\dfrac{\sqrt{x}-3}{x-9} & ;x\neq 9\\ \dfrac{1}{6} & ;x = 9\end{cases}$