

Calculus
Chapter 2 Review Solutions

I have done my best to make sure all the solutions are correct. Inevitably, there seem to be typos. If you do not agree/understand a solution, email me or find time to ask me about them BEFORE the exam.

1. Use direct substitution. $\lim_{x \rightarrow \frac{5}{2}} \lfloor x \rfloor = \lfloor \frac{5}{2} \rfloor = 2$

2. The numerator has a higher degree than the denominator, so $\lim_{x \rightarrow \infty} \frac{x^2 + 5x - 3}{3x + 2}$ does not exist (DNE).

3. The numerator and denominator have the same degree, so $\lim_{x \rightarrow \infty} \frac{x^2 + 5x - 3}{3x^2 + 2} = \frac{1}{3}$

4. The numerator has a lower degree than the denominator, so $\lim_{x \rightarrow \infty} \frac{x^2 + 5x - 3}{3x^3 + 2} = 0$

5. Multiply by $\frac{2}{2}$... $\lim_{x \rightarrow 0} \frac{x}{\sin(2x)} = \lim_{x \rightarrow 0} \frac{2x}{2\sin(2x)} = \frac{1}{2}$

6. Take out the $\frac{1}{2}$ to get $\lim_{x \rightarrow \infty} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x \rightarrow \infty} \frac{\sin x}{x} = \frac{1}{2} \cdot 0 = 0$

7. Rewrite $\tan(5x)$... $\lim_{x \rightarrow 0} \frac{\tan(5x)}{\sin(3x)} = \lim_{x \rightarrow 0} \frac{\sin(5x)/\cos(5x)}{\sin(3x)} = \lim_{x \rightarrow 0} \frac{\sin(5x)}{\cos(5x) \cdot \sin(3x)}$

Multiply by $\frac{5x}{5x}$... $\lim_{x \rightarrow 0} \frac{\sin(5x) \cdot 5x}{5x \cdot \cos(5x) \cdot \sin(3x)} = \lim_{x \rightarrow 0} \frac{\sin(5x)}{5x} \cdot \frac{5x}{\cos(5x) \cdot \sin(3x)}$

Multiply by $\frac{3}{3}$, rearrange the terms and take out $\frac{5}{3}$... $\lim_{x \rightarrow 0} \frac{\sin(5x)}{5x} \cdot \frac{5x \cdot 3}{\cos(5x) \cdot \sin(3x) \cdot 3} = \frac{5}{3} \lim_{x \rightarrow 0} \frac{\sin(5x)}{5x} \cdot \frac{3x}{\sin(3x)} \cdot \frac{1}{\cos(5x)}$

Now you can evaluate the limit ... $\frac{5}{3} \lim_{x \rightarrow 0} \frac{\sin(5x)}{5x} \cdot \frac{3x}{\sin(3x)} \cdot \frac{1}{\cos(5x)} = \frac{5}{3} \cdot 1 \cdot \frac{1}{1} = \frac{5}{3}$

8. The numerator has a higher degree than the denominator, so $\lim_{x \rightarrow \infty} \frac{4x^2 + 5x}{x - 3}$ does not exist (DNE).

9. The numerator has the same degree as the denominator, so $\lim_{x \rightarrow \infty} \frac{5x - 7x^2}{4x^2 + 1} = \frac{-7}{4}$

10. A graph of this (make a table if necessary) would show that the $\lim_{x \rightarrow -3^+} \frac{|x+3|}{x+3} = 1$ and $\lim_{x \rightarrow -3^-} \frac{|x+3|}{x+3} = -1$ meaning

$\lim_{x \rightarrow -3} \frac{|x+3|}{x+3}$ does not exist (DNE).

11.

x	1000	10000	100000	1000000	10000000	100000000
$\left(1 + \frac{1}{x}\right)^x$	2.716923932	2.718145927	2.718268237	2.718280469	2.718281694	2.718281786

Recognize the number? It's e ... Add this to your notecards under "Limits you should know". ... yeah ... those things you have to hand in before your test! ☺

12. Since you're approaching 2 from both sides, pick two numbers close to 2 on both sides.

x	$\frac{x+3}{x-2}$
1.99	-499
1.999	-4999
2.001	5001
2.01	501

Based on the table, $\lim_{x \rightarrow 2^-} \frac{x+3}{x-2}$ DNE, and as $x \rightarrow 2^-$, $\frac{x+3}{x-2} \rightarrow -\infty$.

Also, as $x \rightarrow 2^+$, $\frac{x+3}{x-2} \rightarrow \infty$.

For questions 13 and 14, find ALL asymptotes (vertical, horizontal, and oblique) and justify your response.

13. This function has a vertical asymptote at $x = 0$ because as $x \rightarrow 0^+$, $\ln x \rightarrow -\infty$.

14. Since $f(x) = \frac{\cancel{(x+2)}(x-3)}{\cancel{(x+2)}(x-1)} = \frac{x-3}{x-1}$ there is NOT a vertical asymptote at $x = -2$, but there IS a vertical asymptote at

$x = 1$, since $\lim_{x \rightarrow 1^-} \frac{x-3}{x-1}$ DNE because as $x \rightarrow 1^-$, $\frac{x-3}{x-1} \rightarrow \infty$ and as $x \rightarrow 1^+$, $\frac{x-3}{x-1} \rightarrow -\infty$. Also, since

$\lim_{x \rightarrow \infty} \frac{(x+2)(x-3)}{(x+2)(x-1)} = 1$, there is a horizontal asymptote at $y = 1$.

15. Since $h(x) = \frac{\cancel{(x-1)}\cancel{(x+3)}}{\cancel{(x+3)}(x-2)} = \frac{x-1}{x-2}$, there is a vertical asymptote at $x = 2$, but there is a hole when $x = -3$. Therefore,

$\lim_{x \rightarrow c} h(x)$ exists for all real numbers except for $x = 2$.

16.

a) Since $g(x) = \frac{x^2+5x+6}{x^2+3x+2} = \frac{\cancel{(x+2)}(x+3)}{(x+1)\cancel{(x+2)}} = \frac{x+3}{x+1}$, the domain is $x \neq -1$ and $x \neq -2$

b) Since there is a removable discontinuity (a hole) at $x = -2$, the limit as x approaches -2 exists, but since there is a vertical asymptote at $x = -1$, the limit does not exist as x approaches -1 .

c) Since $\lim_{x \rightarrow \infty} \frac{x^2+5x+6}{x^2+3x+2} = 1$, there is a horizontal asymptote at $y = 1$.

d) There is a vertical asymptote at $x = -1$, since $\lim_{x \rightarrow -1^+} \frac{x^2+5x+6}{x^2+3x+2} = \lim_{x \rightarrow -1^+} \frac{x+3}{x+1}$ which does not exist (DNE) and as

$x \rightarrow -1^+$, $\frac{x+3}{x+1} \rightarrow \infty$.

e) The only value of x less than -1 where $g(x)$ is not continuous is $x = -2$.

Since there is a removable discontinuity at $x = -2$, we just need to define the value of $g(-2)$ to fill the hole. Using the

simplified version of $g(x)$, we see that $g(-2) = \frac{-2+3}{-2+1} = -1$. Thus we write $g(x) = \begin{cases} \frac{x^2+5x+6}{x^2+3x+2} & ; x \neq -1 \text{ and } x \neq -2 \\ -1 & ; x = -2 \end{cases}$

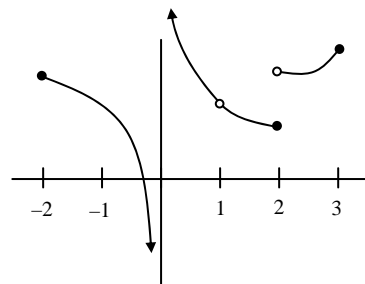
17. Using the function below, over what intervals does $\lim_{x \rightarrow c} f(x)$ exist?

The limit fails to exist at $x = 0$ and at $x = 2$ (both are non-removable discontinuities).

The limit does exist at $x = 1$, even though there is a hole (removable discontinuity).

So, the intervals where $\lim_{x \rightarrow c} f(x)$ exists are $(-2, 0) \cup (0, 2) \cup (2, 3)$.

♫: The $\lim_{x \rightarrow -2^+} f(x)$ also exists, as does $\lim_{x \rightarrow -3^-} f(x)$.



18. Let $y = x^3 - 4x$.

$$\begin{aligned} \text{a) } \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 4(x+h)] - (x^3 - 4x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 4x - 4h - x^3 + 4x}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 4h}{h} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 4) \\ &= 3x^2 + 0 + 0 - 4 \\ &= 3x^2 - 4 \end{aligned}$$

b) Slope at $x = -1$: $3(-1)^2 - 4 = -1$

c) When $x = -1$, $y = 3$ and the slope is -1 : $y - 3 = -1(x + 1)$

d) The slope of the normal line would be $+1$: $y - 3 = 1(x + 1)$

$$\begin{aligned} 19. \lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \\ &= \lim_{h \rightarrow 0} \frac{(4+h) - 4}{h(\sqrt{4+h} + 2)} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{4+h} + 2)} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{4+h} + 2} \\ &= \frac{1}{4} \end{aligned}$$

For questions 20 - 22, find the value of the parameter(s) that would make the function continuous. Justify your response using the definition of continuity.

$$20. j(x) = \begin{cases} ax^2 & ; x < 1 \\ 4x - 2 & ; x \geq 1 \end{cases}$$

In order to be continuous, the only problem occurs when $x = 1$. To be continuous at $x = 1$, $\lim_{x \rightarrow 1} f(x) = f(1)$. In order for

$$\lim_{x \rightarrow 1} f(x) \text{ to exist, } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x).$$

$$a(1)^2 = 4(1) - 2$$

$$a = 2$$

Thus, if $a = 2$, $\lim_{x \rightarrow 1} f(x) = 2 = f(1)$, and $f(x)$ is continuous.

21. In order to be continuous, the only problem occurs when $x = 0$. To be continuous at $x = 0$, $\lim_{x \rightarrow 0} k(x) = k(0)$.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 3x}{x} &= a \\ \lim_{x \rightarrow 0} \frac{\sin 3x}{x} \cdot \frac{3}{3} &= a \\ 3 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} &= a \\ 3 \cdot 1 &= a \\ 3 &= a \end{aligned}$$

Thus, if $a = 3$, $\lim_{x \rightarrow 0} k(x) = 3 = k(0)$, and $k(x)$ is continuous.

22. ~~$f(x) = \begin{cases} a + 3x & ; x \leq -1 \\ bx^2 + 2 & ; x > -1 \end{cases}$... Skip this problem for now ...~~

23. Let $y = \frac{x^2 + 5x - 3}{x - 2}$.

a) End Behavior Model: $\frac{x^2}{x} = x$

b) End Behavior: as $x \rightarrow \infty$, $y \rightarrow \infty$, and as $x \rightarrow -\infty$, $y \rightarrow -\infty$

c) There is a vertical asymptote at $x = 2$. Since the $\lim_{x \rightarrow \infty} y$ does not exist (DNE), there are no horizontal

asymptotes. However, using long division you get $\frac{x^2 + 5x - 3}{x - 2} = x + 7 + \frac{11}{x - 2}$. Therefore, there is a slanted (oblique) asymptote at $y = x + 7$.

$$\begin{aligned} 24. \text{ When } x_0 = 0, \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} &= \lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^{2/5} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h^{3/5}} \end{aligned}$$

This limit does not exist, since as $h \rightarrow 0^+$, $\frac{1}{h^{3/5}} \rightarrow \infty$, and as $h \rightarrow 0^-$, $\frac{1}{h^{3/5}} \rightarrow -\infty$. Thus according to the statement made in the problem, there is a vertical tangent line at $x = 0$ Try looking at a graph to convince yourself of this. (You may want to zoom in close to the origin.)

25. In order for $k(x)$ to be continuous at $x = 9$, we need $\lim_{x \rightarrow 9} k(x) = k(9)$. Since $k(9)$ is currently not defined, we just need to evaluate the limit and then assign the same value to $k(9)$. In order to evaluate the limit, we must rationalize the numerator because if you just plug in $x = 9$, you get $\frac{0}{0}$.

$$\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} = \lim_{x \rightarrow 9} \frac{\cancel{x - 9}}{(\cancel{x - 9})(\sqrt{x} + 3)} = \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{6}$$

Since $k(9) = \frac{1}{6}$, we now have the extended (and continuous) version of $k(x) = \begin{cases} \frac{\sqrt{x} - 3}{x - 9} & ; x \neq 9 \\ \frac{1}{6} & ; x = 9 \end{cases}$