1.1 LINES

Notecards from Section 1.1: Rules for Rounding
Round or Truncate all final answers to 3 decimal places. Do NOT round before you reach your final answer.

Much of Calculus focuses on the concept of "local linearity", meaning that even if a function curves, if you were to pick a point and stay very close (local) to that point, the function behaves very much like that of a line.

Example 1: Graph the functions $y = \sin x$ and $y = x$ on your calculator. Obviously these are not the same function.

However, if you were to stay close to the point (0, 0), these two functions are very close. To see this, use the ZOOM feature of your calculator, and zoom in on (0, 0). Try zooming in more than once.

We can say that as long as we stay "close" to (0, 0), the functions $y = \sin x$ and $y = x$ are almost the same thing. Now, the concept of "close" is more complicated than it might sound, but more on that in chapter 2. For now, we focus on lines.

As stated in the syllabus, calculus has to do with change. For notational purposes, we use the capital Greek letter delta, Δ .

Slope

The slope of a non-vertical line is given by

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

A vertical line has _____, and a horizontal line has _____ .

Parallel Lines have slopes that are _____ .

Perpendicular Lines have slopes that are _____.

IMPORTANT ♪ : You will be best served in calculus if you think of slope as a _____.

The slope between two points will be referred to as _____.

Equations of a Line

The first equation of a line you used in algebra was probably the *slope – intercept form*: _____
        The slope is _____, and the *y*-intercept is _____.

In calculus, it is actually easier to write the equation of a line in *point – slope form*: _____
        The point is _____, and the slope is _____.

♪ : To write an equation of a line, all you need is a _____ and the _____.

Another format used to write the equation of a line is called *standard (general) form*: _____
        All variables are on the same side (usually in alphabetical order).

Example 2: Which of the equations above has "*y* written as a function of *x*" ?

Example 3: The point-slope form is written as _____ if you want "*y* written as a function of *x*"

Example 4: Find the equations of the lines passing through (–2, 4) and having the following characteristics:

    a) Slope of $\frac{7}{16}$

    b) Parallel to the line $5x - 3y = 3$

    c) Passing through the origin

    d) Parallel to the y – axis.

Example 5: Find the equations of the lines passing through (1, 3) and having the following characteristics:

    a) Slope of $-\frac{2}{3}$

    b) Perpendicular to the line $x + y = 0$

    c) Passing through the point (2, 4)

    d) Parallel to the x – axis.

Regression Analysis is a process of finding a curve to fit a set of data. The basic process involves plotting the points and finding a function that "best fits" those points. The curve you find is called the regression curve. For the purposes of this section, our "curve" is linear, but it could be a parabola or other power function, a logarithmic function, a trigonometric function, or an exponential function.

Example 6: The median price of existing single-family homes has increased consistently during the past few years. However, the data in the table below show that there have been differences in various parts of the country.

Year	1999	2000	2001	2002	2003
South ($)	145,900	148,00	155,400	163,400	168,100
West ($)	173,700	196,400	213,600	238,500	260,900

    a) Find the linear regression equation for home cost in the South. _____

    b) What does the slope of the regression line represent?

    c) Find the linear regression equation for home cost in the West. _____

    d) Where is the median price increasing more rapidly, in the South or the West? Explain.

    e) Using your regression equation in part *c*, predict the cost of a home in the West in the year 2011.

1.2 FUNCTIONS AND GRAPHS

Notecards from Section 1.2: Odd/Even Functions

Functions

In the last section we discussed lines and when we needed to write "*y* as a function of *x*". But what is a function?
In Algebra 1, we defined a function as a rule that assigned one and only one (a unique) output for every input. We called the input the *domain* and the output the *range*.

Definition: A **function** from a set *D* to a set *R* is a rule that assigns a *unique* element in *R* to each element in *D*. The vertical line test is the graphical interpretation of this definition.

Example 1: There are 3 domain restrictions you MUST continue to be aware of throughout this course. What are they?

Intervals

In this course we would like not only to know what the domain and range are, but how to describe them with the correct notation. The domain and range of a function could be all real numbers, or we may need to limit the domain and/or the range using *intervals* that are either *open* or *closed*.

Open and closed intervals have endpoints. If the **endpoint is included**, then we say the interval is **closed** at that point, and if the **endpoint is not included,** then we say the interval is **open** at that point. We use a parenthesis to indicate open and a bracket to indicate closed.

Example 2: Use interval notation AND inequality notation to describe each interval on the *x* – axes below.
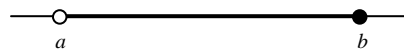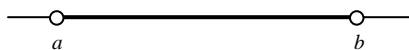
♩: While it is many times possible to just look for the restrictions to the domain, the range of a function is easier to determine if you have a graph or you know what the graph looks like. You should know what the following basic functions look like without having to use your calculator. Can you draw an accurate sketch … at least 3 points on each?

$y = x$
$\qquad y = x^2$
$\qquad y = x^3$
$\qquad y = \sqrt{x}$

$y = \dfrac{1}{x}$
$\qquad\qquad\qquad y = \dfrac{1}{x^2}$

$y = |x|$
$\qquad y = \lfloor x \rfloor$
$\qquad y = ab^x, \quad 0 < b < 1$
$\qquad y = ab^x, \quad 1 < b$

$y = \log_b x, \quad b > 1$
$\qquad y = \sin x$
$\qquad y = \cos x$
$\qquad y = \tan x$

Do you know the domain and range of each of these functions?

Even and Odd Functions

Recognizing the behavior of functions is not limited to their domain and range. Many functions have the symmetric property of being odd or even. You need to be able to recognize the graph of a function as odd or even, AND you need to understand how to show/verify/prove that a function is even or odd algebraically.

Graphical Recognition of Even and Odd Functions

An **EVEN** function is <u>symmetrical about the y – axis</u>. *Example*: $y = \cos x$

An **ODD** function is <u>symmetrical about the origin.</u> *Example*: $y = \sin x$

Algebraic Properties of Even and Odd Functions

An **EVEN** function has the property that $f(-x) = f(x)$.
That is, if you plug in "–x" into the function and simplify, you will obtain the <u>original</u> function.

An **ODD** function has the property that $f(-x) = -f(x)$.
That is, if you plug in "–x" into the function and simplify, you will obtain the <u>opposite of the original</u> function.

Example 3: Prove whether the following functions are even, odd, or neither.

   a) $g(x) = x^3 - x$ b) $h(x) = 1 + \cos x$

Piecewise Functions

Some functions are broken into pieces and behave differently depending on the restricted domain of each piece. Such functions are called piecewise functions. An example of a function that can be written as a piecewise function is the absolute value function $f(x) = |x|$. Be sure to use correct domain restrictions.

Example 4: Sketch $f(x) = |x|$, and write an equation for the two "pieces" using a domain appropriate to each piece.

Example 5: Write a piecewise function for the graph at the right.
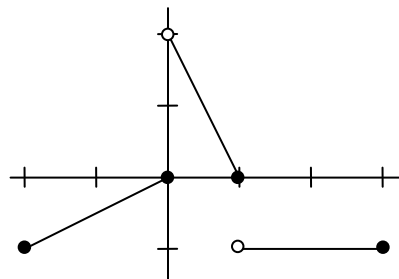
Composite Functions

When the range of one function is used as the domain of a second function we call the entire function a composite function. Because of the range of the first function is used as the domain of the second, you must not assume the final function has the same domain and range as it would have had if written independently.

We use the notation $(f \circ g)(x) = f(g(x))$ to describe composite functions.

This is read as "*f* composed with *g*" or "*f* of *g* of *x*".

Example 6: If $f(x) = 1 - x^2$ and $g(x) = \sqrt{x}$, find $g(f(x))$. What is the domain and range of $g(f(x))$?

Example 7: If $f(x) = \dfrac{2x-1}{x+3}$, and $g(x) = \dfrac{3x+1}{2-x}$, find $f(g(x))$. Based on your answer, how might f and g be related?
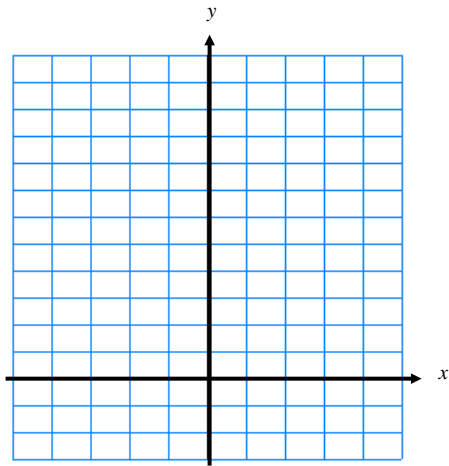
1.3 EXPONENTIAL FUNCTIONS

So far we've dealt with linear functions, piecewise functions, and composite functions. Next up, exponential functions.

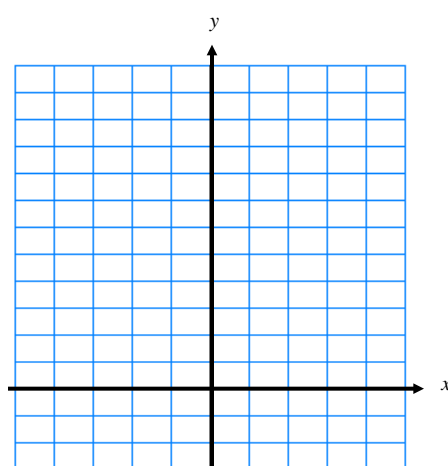*Definition*: *Exponential Function*

If $b > 0$ and $b \neq 1$, then $f(x) = b^x$ is an exponential function with base b.

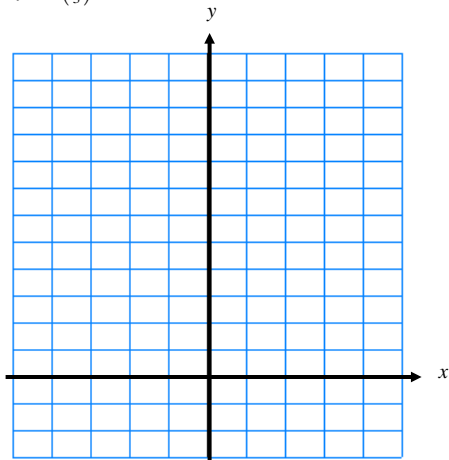*Example 1*: Sketch the following graphs as accurately as possible on the graphs below:
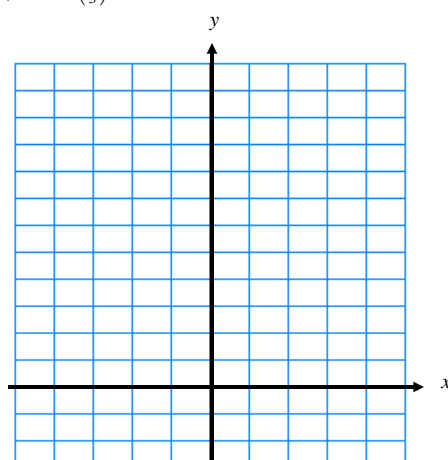
a) $y = 3^x$

b) $y = 2 \cdot 3^x$

c) $y = \left(\frac{1}{3}\right)^x$

d) $y = 2 \cdot \left(\frac{1}{3}\right)^x$

Example 2: Which of these graphs show growth? decay?

Example 3: How does the 2 affect the graph?

Example 4: What is the domain and range of all 4 graphs?

Exponential Growth/Decay Model

In the exponential model $y = a \cdot b^x$, b is the rate of growth if $b > 1$, and b is the rate of decay if $0 < b < 1$.
In either case, the initial value is a.

Example 5: Suppose you invest \$12,000 in an account that earns you 5% interest compounded monthly for 10 years.

      a) What is the initial amount?

      b) What is the growth rate?

      c) How many times does your money grow in 10 years? (How many times is interest added to your account?)

      d) How much money will you have in 10 years?

The Number e

Many exponential functions in the real world (ones that grow/decay on a continuous basis) are modeled using the base of e.
Just like $\pi \approx 3.14$, we say $e \approx 2.718$. We can also define e using the function $\left(1 + \frac{1}{x}\right)^x$ as follows:

$$\text{As } x \to \infty, \ \left(1 + \frac{1}{x}\right)^x \to e$$

Try convincing yourself that this function approaches e using the $\boxed{\text{TABLE}}$ function of your calculator.

Example 6:   Suppose the interest in example 5 was compounded continuously. How much would you have in 10 years?

Example 7: Using your graphing calculator, let $Y_1 = x^2$ and $Y_2 = 2^x$. Graph both equations in the same window.

      a) Solve the equation $x^2 = 2^x$ using your graphing calculator.   Where are the solutions to this equation and how many are there?

      b) Clear the two graphs from the screen and use the equation $x^2 - 2^x = 0$. Solve for x by graphing the left side of this equation. Where are the solutions to this equation and how many are there?

      c) What did you learn from the last two questions?

PARENT FUNCTIONS AND TRANSFORMATIONS

Notecards from Parent Functions and Transformations: Transformations (Parent Functions)

Parent Functions

One of the things we do in calculus is study the behavior of functions. Some of the most basic functions you should be able to recognize and graph <u>without the use of a calculator</u>. You should be able to sketch an accurate graph (3 to 5 EXACT points) of the following parent functions: Check the <u>notecard checklist</u> if you aren't sure.

$y = x$ $y = x^2$ $y = x^3$ $y = \sqrt{x}$

$y = \dfrac{1}{x}$ $y = \dfrac{1}{x^2}$

$y = |x|$ $y = \lfloor x \rfloor$ $y = ab^x,\ 0 < b < 1$ $y = ab^x,\ 1 < b$

$y = \log x$ $y = \sin x$ $y = \cos x$ $y = \tan x$

Transformations

Not only should you be able to graph the parent functions above, but you should be able to graph the transformations of these graphs. Without knowing exact points on the parent function, it will be difficult to transform the parent function!

Example 1: Suppose you are given the function $f(x)$. What effect do *a*, *b*, *c*, and *d* have on original function if your new function is $a \cdot f\big(b(x+c)\big) + d$
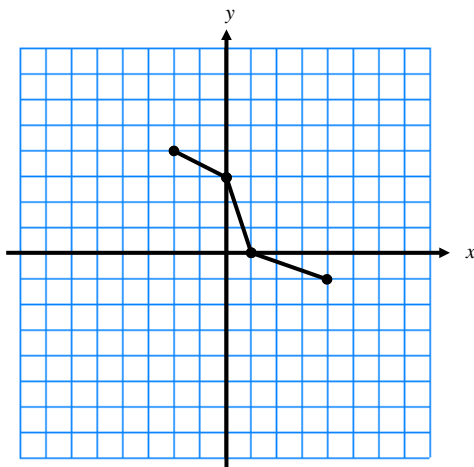
Another way to look at this is with the following chart:

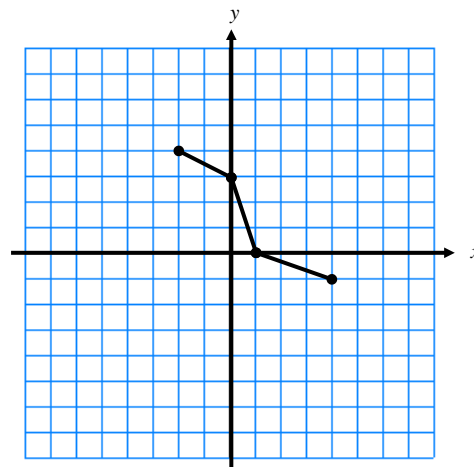|	Inside		Outside	
$+/-$	$f(x+c)$	$f(x-c)$	$f(x)+d$	$f(x)-d$
$\times/\div$	$f(bx)$	$f(-x)$	$a \cdot f(x)$	$-f(x)$

Example 2: Let f be the graph given in the picture below. Graph the following transformations of f.
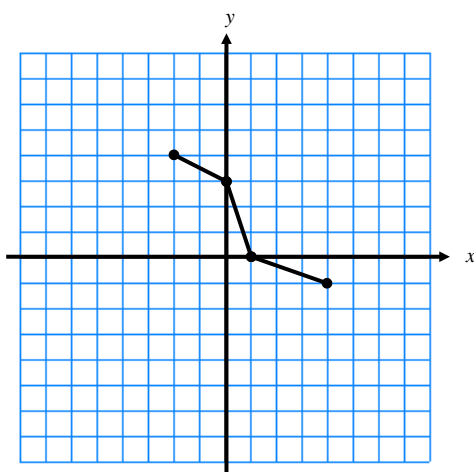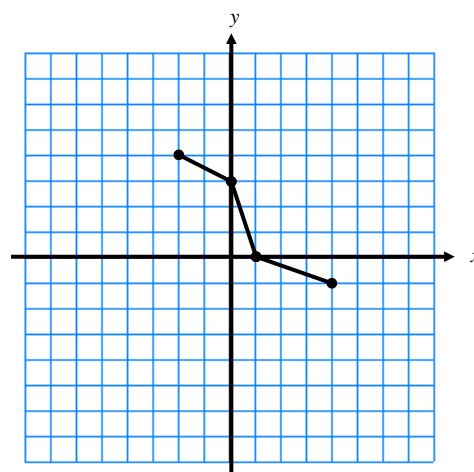
a) $f(x)+2$

b) $-f(x)$

c) $f(x-2)$

d) $f(x+3)$
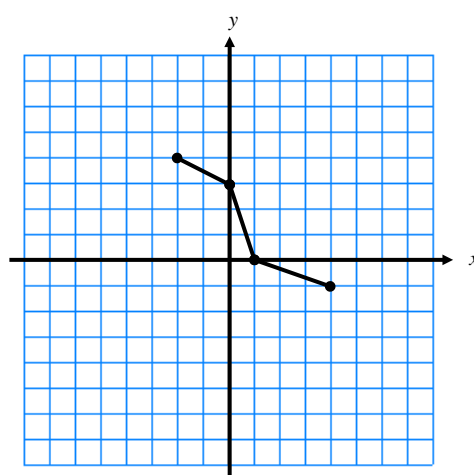
e) $2f(x)$

f) $f(-x)$

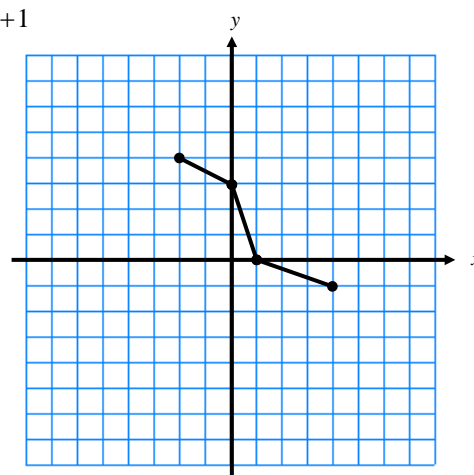g) $f\left(\frac{1}{2}x\right)$

h) $f(2x-6)$

i) $f(|x|)$

j) $|f(x)|$

k) $\frac{1}{2}f(3x-9)+1$

INTRODUCTION TO CONICS

While there are many related ideas and topics that can be used with conics, <u>we are only attempting to scratch the surface</u>. We are only interested in your ability to identify/classify conics by their equation and their graph as well as write an equation of the following conic sections: Circles, Ellipses, and Hyperbolas … you should add these to your parent functions list.

First, what is a conic section? A conic section is the cross-sectional shape formed by slicing a double cone (a cone on top of another cone). Again, we are going to deal with 3 of these … circles, ellipses, and hyperbolas.

Got to the following website for an animated view of these conic sections:
http://mathdemos.gcsu.edu/mathdemos/family_of_functions/conic_gallery.html

General Form of a Conic Section
All conics can be written in the form $Ax^2 + By^2 + Cx + Dy + Exy + F = 0$.
The conics we will be focusing on will have no xy term (unless that is the only term). If $E \neq 0$, the conic will be rotated.
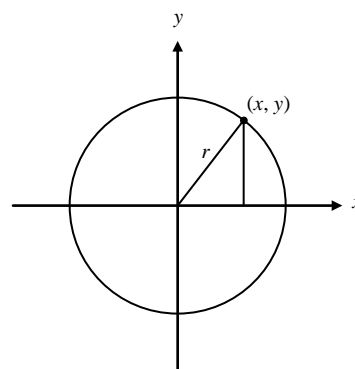
Circles

Consider a circle centered at the origin, with radius r. The Pythagorean Theorem, allows us to generate an equation of the circle to be

$$x^2 + y^2 = r^2 .$$

If we divide both sides of the equation by r^2, we get

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

To graph the circle yourself, find the center, and plot 1 point that is r spaces up, down, left and right of the center.

Example 1: Graph each of the following equations:

a) $x^2 + y^2 = 36$

b) $(x-3)^2 + (y+2)^2 = 25$

Ellipses

An ellipse is simply a circle that has been stretched more in one direction than the other. That "stretch" can be seen in the equation of a circle with the r^2 , where both the horizontal stretch and vertical stretch are r spaces. If the horizontal and vertical stretch are different, the values of each denominator will be different. Thus the general form of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ,$$

where a is the horizontal stretch from the center, and b is the vertical stretch from the center.

To graph the ellipse yourself, find the center, and plot 1 point that is a spaces left and right of the center, and 1 point that is b spaces up and down from the center.
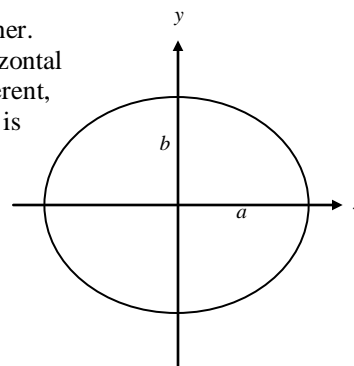
The mathematical definition of an ellipse is the set of all points whose distances from two fixed points (called foci) have a constant sum. Light (or sound) that originates at one focus point inside of the ellipse will reflect off the ellipse to the other focus point.

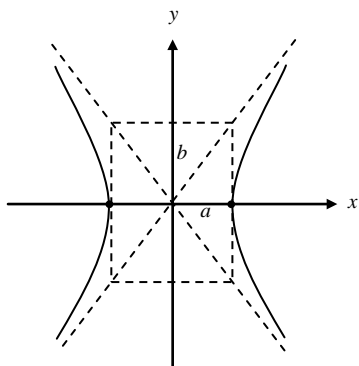*Example 2*: Graph each of the following equations:

a) $\dfrac{x^2}{25} + \dfrac{y^2}{36} = 1$

b) $\dfrac{(y-1)^2}{16} + \dfrac{(x+3)^2}{4} = 1$

Hyperbolas

The mathematical definition of a hyperbola is the set of all point whose distances from two fixed points (called foci) have a constant difference. Thus, the equation for a hyperbola is very similar to an ellipse, except you are subtracting instead of adding. Since the order in which you subtract matters, the hyperbola opens in the direction of whichever axis comes first.

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ … opens left and right   OR $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ … opens up and down
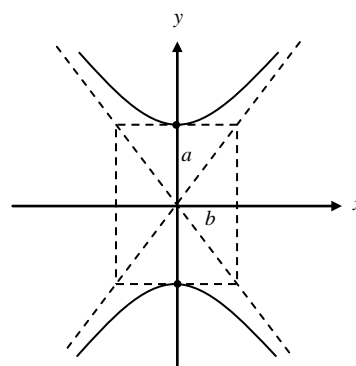
To graph a hyperbola yourself, find the "center", create a rectangular box. Whatever the square root of the denominator is below x, that is the distance you move left and right of the "center". Similarly, the square root of the denominator below y dictates how far you move up and down from the "center". A hyperbola has slanted asymptotes that go through the corners of your rectangular box. If x comes first (meaning you are subtracting y), your hyperbola will open left and right, but if y comes first then your hyperbola will open up and down. The only "points" you need to show will be the "vertices" which are actually on the rectangular box.

For applications of these conic sections or more examples, see Appendix Section A5 in your textbook starting on page 578.

Example 3: Graph each of the following equations:

a) $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$

b) $(y-4)^2 - \dfrac{(x+1)^2}{16} = 1$

1.4 PARAMETRIC EQUATIONS

Notecards from Section 1.4: Trigonometric Identities

Up to this point all the functions we have been looking at have used a single equation with two variables, x and y. In this section we use a third variable to represent the curve. This third variable is called a parameter.

Example 1: Using your graphing calculator, set your window to $X_{min} = -5$, $X_{max} = 80$, $X_{scl} = 5$,
$$Y_{min} = -5, \quad Y_{max} = 20, \quad Y_{scl} = 5.$$

Graph $y = -\dfrac{x^2}{72} + x$, which models the path of an object thrown into the air at a 45° angle at an initial velocity of 48 feet per second. (Just take my word for it ☺)

Use ⌊TRACE⌋ to locate a few points on the graph. What do you learn about the object from this information?

Example 2: Now change your calculator to *parametric* mode and plot $x_1 = 24\sqrt{2}t$.
$$y_1 = -16t^2 + 24\sqrt{2}t$$

Notice it is the same graph … more on that later.

Use ⌊TRACE⌋ to locate a few points on the graph. What do you learn about the object from this information that you did not know before?

Definition: *Parametric Curve*

If x and y are given as functions
$$x = f(t), \quad y = g(t)$$
over an interval of t values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a parametric curve.
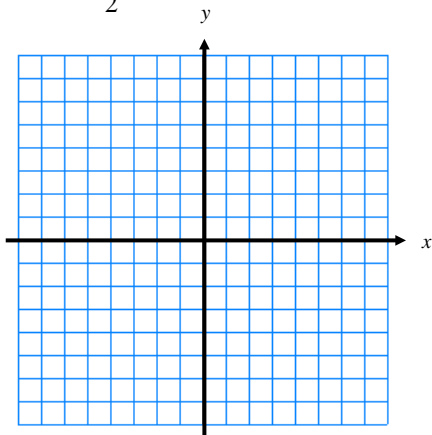

Graphing Parametric Curves Without a Calculator

Just like when you learned to graph for the first time back in Algebra 1, we are going to make a table of values. The difference is that we now have three variables instead of two.

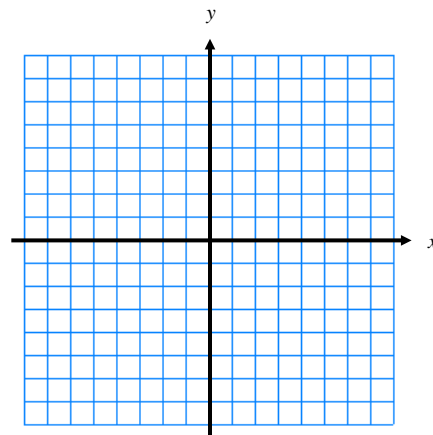*Example 3*: Graph the parametric curve
$$x = t^2 - 4 \qquad -2 \le t \le 3$$
$$y = \frac{t}{2}$$

Example 4: Graph the parametric curve
$$x = 4t^2 - 4 \qquad -1 \le t \le \tfrac{3}{2}$$
$$y = t$$

*Example 5*: Compare and contrast the two graphs above.

Changing from Parametric to Rectangular (Cartesian)

To change a parametric equation back into a more familiar rectangular (Cartesian) equation you must eliminate the parameter. The typical approach to doing this is to solve for the parameter in one of the equations and then simply substitute that solution into the other equation.

Example 6: Change the parametric equation defined by $x = \sqrt{t}$ into a Cartesian equation.
$$y = 2 - t$$

The next two examples illustrate another way to eliminate the parameter by using trigonometric identities.

Example 7: Change the parametric equation $x = 3\cos\theta + 2$ into a Cartesian equation and then graph .
$$y = 5\sin\theta - 1$$
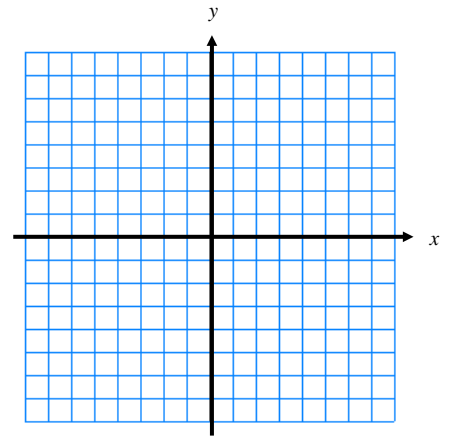
Example 8: Change the parametric equation defined by $x = 4\sec\theta$ into a Cartesian equation and then graph.
$$y = 3\tan\theta$$

Writing a Parametric Equation

Changing from Rectangular to Parametric means you get to create a parameter. As the graphs on the previous page indicated, your choice of parameter should not change the shape of the graph, only the "speed" in which the graph is drawn.

Example 9: Find a parametrization for the left half of the parabola $y = x^2 + 2x$.

Example 10: Find a parametrization for the line segment with endpoints (–1, 3) and (3, –2).

1.5 FUNCTIONS AND LOGARITHMS

Notecards from Section 1.5: Finding and Proving Inverse Functions

Inverse Functions

In technical jargon, an inverse of a function maps the elements of the range to the elements of the domain. In English, this means that the inverse of a function reverses the domain and range. Not all graphs were defined as functions, and we had the *vertical line test* to determine whether a graph was or was not a function. Similarly, not all functions have an inverse that is a function, and we have the *horizontal line test* to determine whether or not a given function has an inverse function.

Definition: *One – to – One Function*

A function $f(x)$ is **one – to – one** on a domain D if $f(a) \neq f(b)$ whenever $a \neq b$.
A function that is one – to – one has an inverse.

The definition above can be seen graphically with the use of a horizontal line test. If there are two x – values for any given y – value of function, then the function does NOT have an inverse.

Example 1: Does $y = x^2 + 5x$ have an inverse? Why or why not?

Example 2: Does $y = x^3 + x$ have an inverse? Why or why not?

Once we know whether a function has an inverse, our next task is to find an equation and/or a graph for the inverse.

Finding *the Inverse Graphically (two ways)*

       1. Reflect the graph of the original function over the line $y = x$.
       2. Plot the reverse of the coordinates.

Finding *the Inverse Algebraically*

       Switch the x and y in the original equation, then solve the new equation for y in order to write y as a function of x.

Example 3: Let $f(x) = x^3 - 1$.

       a) Graph the function on the grid to the right.

       b) Draw the line $y = x$

       c) Reflect the graph of $f(x)$ over the line $y = x$.

       d) ***Find*** the inverse of the function algebraically.
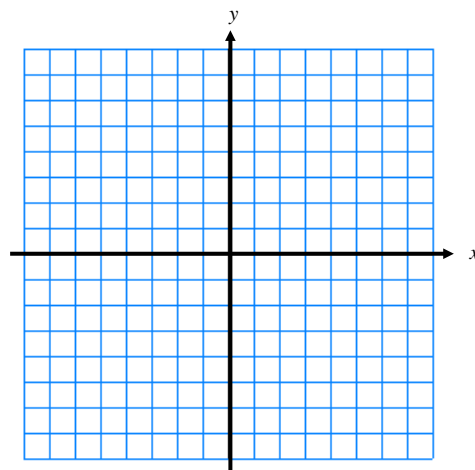
       e) Use your graphing calculator to verify your answer to part *d.*

Verifying/Proving Inverses

It is one thing to **find** the inverse function (either graphically or algebraically), but it is another to **verify** that two functions are actually inverses. Whenever you are verifying anything in mathematics, you must go back and use the definition.

Definition: Inverse Function

A function $f(x)$ has an inverse $f^{-1}(x)$ if and only if $f\left(f^{-1}(x)\right) = x = f^{-1}\left(f(x)\right)$

Example 4: According to this definition, how many composite functions must you use to check whether or not two functions are inverses of each other?
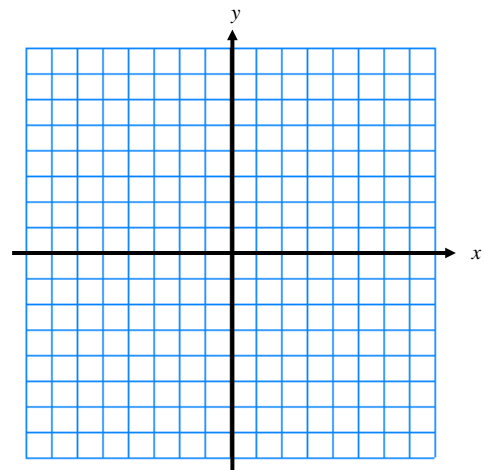
Example 5: Find $f^{-1}(x)$ and verify if $f(x) = \dfrac{x+3}{x-2}$.

Logarithmic Functions

How do Logarithms fit into this discussion? A logarithmic function is just the inverse of an exponential function.

Example 6: Graph $y = 2^x$ and find the inverse of the function graphically.

   The equation of the inverse function is _____.

Properties of Logarithms

Definition of a logarithm:
$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

Inverse Properties of a logarithm:
$$a^{\log_a x} = x \qquad \log_a a^x = x$$

Logarithm of a product:
$$\log_a (xy) = \log_a x + \log_a y$$

Logarithm of a power:
$$\log_a (x^y) = y \cdot \log_a x$$

Logarithm of a quotient:
$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

Change of Base Formula:
$$\log_a x = \frac{\ln x}{\ln a}$$

Other properties:
$$\log_a a = 1 \qquad \log_a 1 = 0$$

Example 7: Evaluate the following without using your calculator.

a) $\log_2 \frac{1}{8} =$

b) $\log_{27} 9 =$

Example 8: Solve for x in the following equations.

a) $\log_2 (x-1) = 5$

b) $3(5^{x-1}) = 86$

c) $\log(8x) - \log(1+\sqrt{x}) = 2$

1.6 TRIGONOMETRIC FUNCTIONS

There are two common measures of angles: degrees and radians. As you'll see, almost all of calculus uses radians. Before beginning any exercise with trigonometric functions, make sure your calculator is set in *radian* mode. (ESPECIALLY THOSE YOU WHO HAD PHYSICS YESTERDAY!) Unless otherwise stated, the angles in the text are measured in radians. For example, sin 3 means the sine of 3 radians, but sin 3° means the sine of 3 degrees. Just for fun, you should understand exactly what a radian is.

Definition: Radian

An angle of **1 radian** is defined to be the angle at the center of a unit circle which spans an arc of length 1, measured counterclockwise.
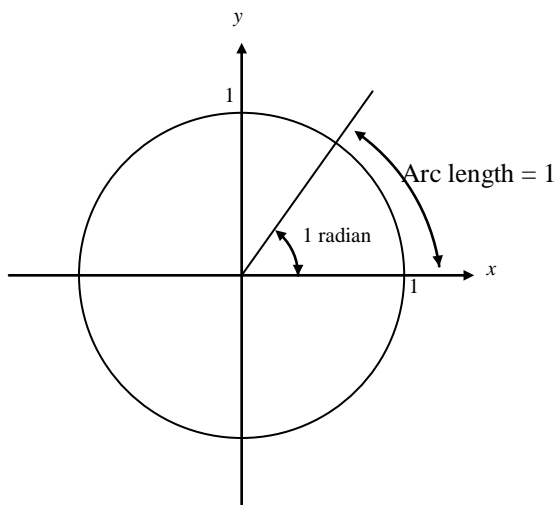
♫ : You should also be VERY familiar with the 6 trigonometric values of the key points on the ***unit circle*** $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \ldots$

Graphs of Trigonometric Functions

Graph one period of all 6 trigonometric functions below. Label at least 5 points (or asymptotes) for each graph.

$y = \sin x$ $y = \cos x$ $y = \tan x$

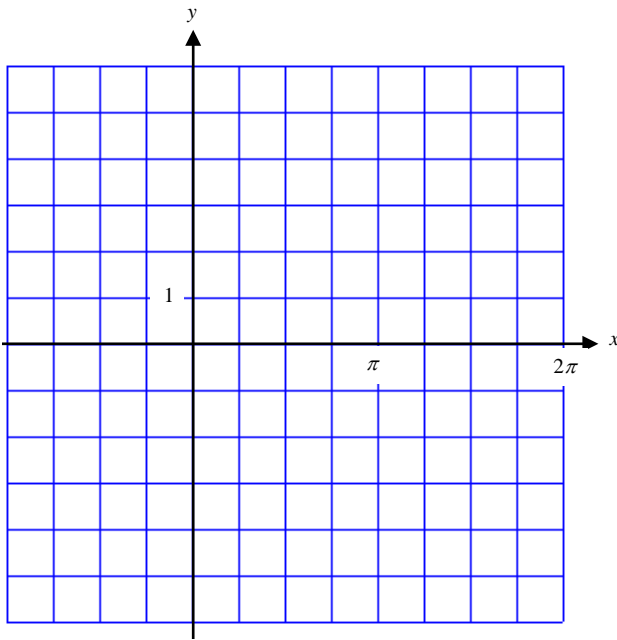$y = \csc x$ $y = \sec x$ $y = \cot x$

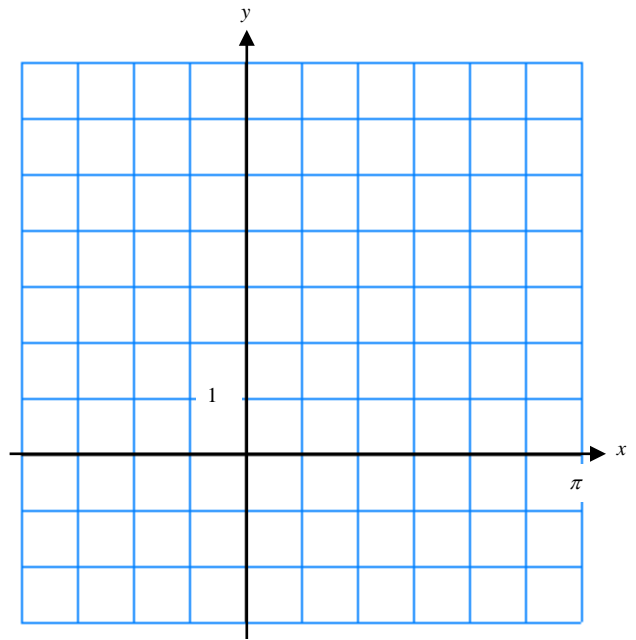*Transformations of Trigonometric Functions*

Example: For each of the following examples, do three things.
      1. Describe the transformation.
      2. Graph the function.
      3. Where appropriate, give the amplitude and the period.

a) $y = \sin(-2x + \pi)$ b) $y = -2\tan(3x - \pi) + 1$

Inverse Trigonometric Functions

Due to the periodic property of trigonometric functions, all 6 of the trig functions fail the horizontal line test. Use the graph of the original function and highlight the portion of the graph used to graph the inverse. Use this highlighted portion to determine the domain and rang e of each inverse function.

$y = \cos x$

$y = \sin x$

Domain of Inverse:

Range of Inverse:

Domain of Inverse:

Range of Inverse:

$y = \tan x$ $y = \cot x$

| Domain of Inverse: | | Domain of Inverse: |
| Range of Inverse: | | Range of Inverse: |

$y = \csc x$ $y = \sec x$

| Domain of Inverse: | | Domain of Inverse: |
| Range of Inverse: | | Range of Inverse: |

Check your answers with the information on page 50 and 51 of your textbook. You can also find graphs of the inverse trig functions on these pages as well.

The restricted domain of the inverse trig functions means you must pay close attention to your solutions. Your calculator only gives you the solution for which the domain of the inverse function is defined. Your calculator also only has 3 of the six inverse functions.

Example: Find the domain and range of the following functions:

a) $\sin\left(\cos^{-1}(x)\right)$ b) $\sec\left(\tan^{-1}(x)\right)$

Example: Evaluate the expression WITHOUT a calculator. $\tan\left(\sin^{-1}\left(\dfrac{12}{13}\right)\right)$

Example: Solve for x: $\sec x = 3$ where $0 \le x \le 2\pi$.