AP Calculus
Chapter 6 Review

Name: KEY
Block: ______

For questions 1 – 21, integrate each of the following indefinite integrals.

1. $\int \frac{3x}{\sqrt[3]{x^2+3}}\,dx$

Let $u = x^2+3$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$\frac{3}{2}\int \frac{du}{\sqrt[3]{u}}$

$= \frac{3}{2}\int u^{-1/3}du$

$= \frac{3}{2}\left(\frac{3}{2}u^{2/3}\right) + C$

$\boxed{= \frac{9}{4}(x^2+3)^{2/3} + C}$

2. $\int \sec(5x)\,dx$

Let $u = 5x$
$du = 5\,dx$
$\frac{1}{5}du = dx$

$\frac{1}{5}\int \sec(u)\,du$

$= \frac{1}{5}\ln|\sec(u) + \tan(u)| + C$

$\boxed{= \frac{1}{5}\ln|\sec(5x) + \tan(5x)| + C}$

3. $\int \frac{x-1}{x+1}\,dx$

$$\begin{array}{r|l} & 1 \\ x+1 & x-1 \\ & -(x+1) \\ \hline & -2 \end{array}$$

$= \int\left(1 - \frac{2}{x+1}\right)dx$

$= \int 1\,dx - \int \frac{2}{x+1}\,dx$

$2\int \frac{du}{u} = 2\ln|u|$

Let $u = x+1$
$du = dx$

$= x$

$\boxed{= x - 2\ln|x+1| + C}$

4. $\int 9\sin x\,dx = \boxed{-9\cos x + C}$

5. $\int 7^{3x}\,dx$

Let $u = 3x$
$du = 3dx$
$\frac{1}{3}du = dx$

$\frac{1}{3}\int 7^u du$

$= \frac{1}{3}\cdot\frac{7^u}{\ln 7} + C$

$\boxed{= \frac{1}{3}\cdot\frac{7^{3x}}{\ln 7} + C}$

6. $\int 2\tan(x)\,dx$

$\boxed{-2\ln|\cos x| + C}$

7. $\int \cot^2 x\,dx$

$\boxed{= -\cot x - x + C}$

8. $\int \frac{(x+1)^2}{x^{1/3}}\,dx = \int \frac{x^2+2x+1}{x^{1/3}}\,dx$

$= \int\left(\frac{x^2}{x^{1/3}} + \frac{2x}{x^{1/3}} + \frac{1}{x^{1/3}}\right)dx$

$= \int\left(x^{5/3} + 2x^{2/3} + x^{-1/3}\right)dx$

$\boxed{= \frac{3}{8}x^{8/3} + 2\cdot\frac{3}{5}x^{5/3} + \frac{3}{2}x^{2/3} + C}$

9. $\int \cos^2(8x)\,dx = \frac{1}{8}\int \cos^2(u)\,du$

Let $u = 8x$
$du = 8\,dx$
$\frac{1}{8}du = dx$

10. $\int \frac{dx}{4+9x^2} = \frac{1}{4}\int \frac{dx}{1+\frac{9x^2}{4}}$

Let $u = \frac{3x}{2}$
$du = \frac{3}{2}dx$
$\Rightarrow \frac{2}{3}du = dx$

$\frac{1}{4}\cdot\frac{2}{3}\int \frac{du}{1+u^2}$

$= \frac{1}{6}\tan^{-1}(u) + C$

$\boxed{= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + C}$

11. $\int\left(10\cos t + \sin^2(10t)\right)dt$

$10\int \cos t\,dt + \int \sin^2(10t)\,dt$

$= 10\sin t +$

Let $u = 10t$
$du = 10dt$
$\frac{1}{10}du = dt$

$\frac{1}{10}\int \sin^2(u)\,du$

$= 10\sin t + \frac{1}{10}\left[\frac{1}{2}u - \frac{1}{4}\sin(2u)\right] + C$

$\boxed{= 10\sin t + \frac{1}{2}t - \frac{1}{40}\sin(20t) + C}$

12. $\int \sec^2 x\,dx = \boxed{\tan x + C}$

13. $\int \frac{dx}{x\sqrt{x^2-4}}$

$\int \frac{dx}{x\sqrt{4\left(\frac{x^2}{4}-1\right)}} = \int \frac{dx}{2x\sqrt{\frac{x^2}{4}-1}}$

Let $u = \frac{x}{2} \Rightarrow 2u = x$
$du = \frac{1}{2}dx \Rightarrow 2du = dx$

$\int \frac{\cancel{2}du}{\cancel{2}(2u)\sqrt{u^2-1}} = \frac{1}{2}\int \frac{du}{u\sqrt{u^2-1}} = \frac{1}{2}\sec^{-1}(u) + C$

$\boxed{= \frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right) + C}$

14. $\int 5e^{3x}\cot\left(e^{3x}\right)dx$

Let $u = e^{3x}$
$du = e^{3x}\cdot 3dx$
$\Rightarrow \frac{1}{3}du = e^{3x}dx$

$\frac{5}{3}\int \cot(u)\,du$

$= \frac{5}{3}\ln|\sin(u)| + C$

$\boxed{= \frac{5}{3}\ln|\sin(e^{3x})| + C}$

15. $\int \frac{1}{\sec(12x)}\,dx = \int \cos(12x)\,dx$

Let $u = 12x$
$du = 12dx$
$\Rightarrow \frac{1}{12}du = dx$

$= \frac{1}{12}\int \cos(u)\,du$

$= \frac{1}{12}\sin(u) + C$

$\boxed{= \frac{1}{12}\sin(12x) + C}$

16. $\int 7\csc\left(\frac{x}{4}\right)dx$

Let $u=\frac{x}{4}$

$du=\frac{1}{4}dx$

$\Rightarrow 4du=dx$

$7\cdot 4\int \csc(u)\,du$

$=28\ln|\csc(u)-\cot(u)|+c$

$\boxed{=28\ln\left|\csc\left(\frac{x}{4}\right)-\cot\left(\frac{x}{4}\right)\right|+C}$

17. $\int \tan^2\left(\frac{x}{5}\right)dx$

Let $u=\frac{x}{5}$

$du=\frac{1}{5}dx$

$\Rightarrow 5du=dx$

$5\int \tan^2(u)\,du$

$=5\left[\tan(u)-u\right]+C$

$=5\tan\left(\frac{x}{5}\right)-5\left(\frac{x}{5}\right)+C$

$\boxed{=5\tan\left(\frac{x}{5}\right)-x+C}$

18. $\int \frac{x}{x^2-4}dx$

Let $u=x^2-4$

$du=2x\,dx$

$\frac{1}{2}du=x\,dx$

$\frac{1}{2}\int\frac{du}{u}$

$=\frac{1}{2}\ln|u|+C$

$\boxed{=\frac{1}{2}\ln|x^2-4|+C}$

19. $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$

Let $u=e^x$

$du=e^x dx$

$=\int\frac{du}{\sqrt{1-u^2}}$

$=\sin^{-1}(u)+C$

$\boxed{=\sin^{-1}(e^x)+C}$

20. $\int \csc^2 x\,dx=\boxed{-\cot x+C}$

21. $\int xe^{-x^2+3}dx$

Let $u=-x^2+3$

$du=-2x\,dx$

$\Rightarrow -\frac{1}{2}du=x\,dx$

$-\frac{1}{2}\int e^u du$

$=-\frac{1}{2}e^u+C$

$=\boxed{-\frac{1}{2}e^{-x^2+3}+C}$

For questions 22 – 26, evaluate each definite integral without a calculator. Check your answer with your calculator.

22. $\int_{-1}^{1}(x^2-5)^2\,dt=\int_{-1}^{1}(x^4-10x^2+25)\,dx$

$=\frac{1}{5}x^5-\frac{10}{3}x^3+25x\Big|_{-1}^{1}$

$=\left[\frac{1}{5}(1)^5-\frac{10}{3}(1)^3+25(1)\right]-\left[\frac{1}{5}(-1)^5-\frac{10}{3}(-1)^3+25(-1)\right]$

$=\frac{1}{5}-\frac{10}{3}+25-\left(-\frac{1}{5}+\frac{10}{3}-25\right)=\frac{2}{5}-\frac{20}{3}+50$

$=\frac{6-100+750}{15}$

$=\boxed{\frac{656}{15}}$

$\begin{array}{r} {}^{3}5 \\ 50 \\ \hline 750 \end{array}$

23. $\int_{-4}^{-2}\frac{dx}{x^2+6x+10}=\int_{-4}^{-2}\frac{dx}{x^2+6x+\underline{9}+10-\underline{9}}$

$=\int_{-4}^{-2}\frac{dx}{(x+3)^2+1}$

Let $u=x+3$

$du=dx$

$=\int_{-1}^{1}\frac{du}{u^2+1}=\tan^{-1}(u)\Big|_{-1}^{1}$

$=\tan^{-1}(1)-\tan^{-1}(-1)$

$=\pi/4-(-\pi/4)$

$=\frac{2\pi}{4}=\boxed{\pi/2}$

24. $\int_0^1\frac{3+x}{x^2+1}dx=\int_0^1\frac{3}{x^2+1}dx+\int_0^1\frac{x}{x^2+1}dx$

Let $u=x^2+1$

$du=2x\,dx$

$\frac{1}{2}du=x\,dx$

$=3\tan^{-1}(x)\Big|_0^1+\frac{1}{2}\int_1^2\frac{du}{u}$

$=\left[3\tan^{-1}(1)-3\tan^{-1}(0)\right]+\frac{1}{2}\ln|u|\Big|_1^2$

$=3\left(\frac{\pi}{4}\right)-3(0)+\left[\frac{1}{2}\ln(2)-\frac{1}{2}\ln(1)\right]$

$\boxed{=3\pi/4+\frac{1}{2}\ln 2}$

25. $\int_0^6\frac{dx}{7-x}$

Let $u=7-x$

$du=-dx$

$-du=dx$

$=-\int_7^1\frac{du}{u}$

$=-\ln|u|\Big|_7^1$

$=-\ln(1)-(-\ln(7))$

$=\boxed{\ln 7}$

26. $\int_4^9\sqrt{x}(3-4x)\,dx=\int_4^9 x^{1/2}(3-4x)\,dx$

$=\int_4^9\left(3x^{1/2}-4x^{3/2}\right)dx=3\cdot\frac{2}{3}x^{3/2}-4\cdot\frac{2}{5}x^{5/2}\Big|_4^9$

$=\left[2(9)^{3/2}-\frac{8}{5}(9)^{5/2}\right]-\left[2(4)^{3/2}-\frac{8}{5}(4)^{5/2}\right]$

$=\left[54-\frac{1944}{5}\right]-\left[16-\frac{256}{5}\right]$

$=38-\frac{1688}{5}=\boxed{\frac{-1498}{5}}$

27. Derive (SHOW EVERY STEP) $y = y_0 e^{kt}$ from $\frac{dy}{dt} = ky$ and $y(0) = y_0$.

$\frac{dy}{y} = k\,dt$

$\int \frac{dy}{y} = \int k\,dt$

$\ln|y| = kt + C$

$|y| = e^{kt + C}$

$|y| = e^{kt} \cdot e^{C}$

$|y| = Ce^{kt}$

$y = \pm Ce^{kt}$

IF $y(0) = y_0$

then

$y_0 = \pm Ce^{k(0)}$

$y_0 = \pm C$

$\therefore y = y_0 e^{kt}$

28. Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f the slope is given by $\boxed{\frac{3x^2+1}{2y} = dy/dx}$

NOTE $y \neq 0$ which means y is defined for $(-\infty, 0)$ OR $(0, \infty)$

since $y = 4$ is given then y is only defined on $(0, \infty)$.

a) Find the slope of the graph of f at the point where $x = 1$. $(y = 4)$

Slope @ $(1,4)$ $= dy/dx\Big|_{(1,4)} = \frac{3(1)^2+1}{2(4)} = \frac{4}{8} = 1/2$

b) Write an equation for the line tangent to the graph of f at $\underset{y=4}{x = 1}$ and use it to approximate $f(1.2)$

Point $(1,4)$

Slope $= 1/2$

$\boxed{y - 4 = \frac{1}{2}(x-1)}$

$f(1.2) \approx \frac{1}{2}(1.2 - 1) + 4 = \frac{1}{2}(.2) + 4 = \boxed{4.1}$

c) Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx} = \frac{3x^2+1}{2y}$ with the initial condition $f(1) = 4$.

$\int 2y\,dy = \int (3x^2+1)\,dx$

$y^2 = x^3 + x + C$

when $x = 1, y = 4$...

$4^2 = 1^3 + 1 + C$

$14 = C$

$\therefore y^2 = x^3 + x + 14$

$\Rightarrow y = \pm\sqrt{x^3+x+14}$, but since y is defined on $(0,\infty)$ only

$f(x) = \sqrt{x^3+x+14}$

d) Use your solution from part c to find $f(1.2)$

$f(1.2) = \sqrt{(1.2)^3 + (1.2) + 14} = \sqrt{16.928} \approx 4.114$ (close to your approximation in part b)

29. If $\frac{dy}{dx} = y\sec^2 x$ and $\boxed{y = 5 \text{ when } x = 0}$ then $y =$

A $e^{\tan x} + 4$

B $e^{\tan x} + 5$

C $\boxed{\text{C} \quad 5e^{\tan x}}$

D $\tan x + 5$

E $\tan x + 5e^{x}$

$\int \frac{dy}{y} = \int \sec^2 x\,dx$

y is on $(\cancel{-\infty, 0})$ OR $\boxed{(0,\infty)}$ since $y = 5$

$\ln|y| = \tan x + C$

$e^{\tan x + C} = |y|$ since y is defined on $(0,\infty)$, $|y| = y$

$e^{\tan x} \cdot e^{C} = y$

$Ce^{\tan x} = y$

when $x = 0, y = 5$

$Ce^{\tan(0)} = 5$

$\therefore C = 5$

Thus, $y = 5e^{\tan x}$

30. [No Calculator] If $\underbrace{\frac{dy}{dt} = -2y}$ and if $y = 1$ when $t = 0$, what is the value of t for which $y = \frac{1}{2}$?

A) $-\frac{1}{2}\ln 2$

B) $-\frac{1}{4}$

$\boxed{\text{C)} \quad \frac{1}{2}\ln 2}$

D) $\frac{\sqrt{2}}{2}$

E) $\ln 2$

you can solve this . . .

$$\int \frac{dy}{y} = \int -2dt$$

$$\ln|y| = -2t + c$$

$$|y| = e^{-2t+c}$$

$$|y| = e^{-2t}e^{c}$$

$$|y| = Ce^{-2t}$$

y is on ~~$(-\infty,0)$~~ or $(0,\infty)$ since $y = 1$ is given

y is only on $(0,\infty)$ $\Rightarrow$ $|y| = y$

$$\therefore y = Ce^{-2t}$$

when $t = 0, y = 1$

$$1 = Ce^{-2(0)}$$

$$1 = C$$

$$\therefore y = e^{-2t}$$

when $y = 1/2$, $\frac{1}{2} = e^{-2t}$

$$\ln(\tfrac{1}{2}) = -2t$$

$$-\tfrac{1}{2}\ln(\tfrac{1}{2}) = t$$

$$-\tfrac{1}{2}\ln(2^{-1}) = t$$

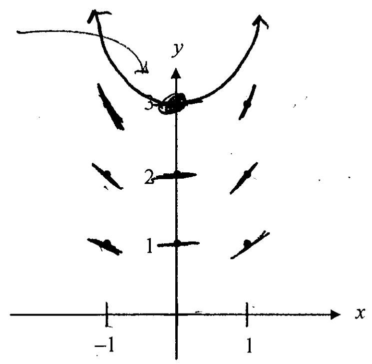
$$+\tfrac{1}{2}\ln(2) = t$$

31. Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$. $\Rightarrow$ $\frac{dy}{y} = \frac{1}{2}x\,dx$

y is on $(-\infty,0)$ OR $(0,\infty)$

a) On the axes provided below, sketch a slope field for the given differential equation at the nine points indicated.

b) Draw a particular solution if $f(0) = 3$



(−1,3)	−3/2
(−1,2)	−1
(−1,1)	−1/2
(0,3)	0
(0,2)	0
(0,1)	0
(1,3)	3/2
(1,2)	1
(1,1)	1/2

c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$.

$y=3$ means y is only defined on $(0,\infty)$

$\therefore |y| = y$

$$\int \frac{dy}{y} = \int \frac{1}{2}x\,dx$$

$$\ln|y| = \frac{1}{4}x^2 + C$$

$$|y| = e^{\frac{1}{4}x^2 + C}$$

$$|y| = e^{1/4x^2} \cdot e^{c}$$

$$|y| = Ce^{1/4x^2}$$

$$y = Ce^{1/4x^2}$$

when $x=0, y=3$

$$3 = Ce^{1/4(0)^2}$$

$$3 = C$$

$$\boxed{\therefore y = 3e^{1/4x^2}}$$

★ use your calculator to compare this graph to the one you drew in part (c).

$$\boxed{\begin{array}{l} y = 3e^{1/4(.2)^2} \\ y \approx 3.030 \end{array}}$$

32. [Calculator] During a certain epidemic, the number of people that are infected at any time increases at rate proportional to the number of people that are infected at that time. If 1,000 people are infected when the epidemic is first discovered, and 1,200 are infected 7 days later, how many people are infected 12 days after the epidemic is first discovered?

Let N = # of people infected

$$\frac{dN}{dt} = K \cdot N$$

Solution $\Rightarrow$

$$N = N_0 e^{kt}$$

$N_0 = 1000$

$$N = 1000e^{kt}$$

$$1200 = 1000e^{k(7)}$$

$$1.2 = e^{7k}$$

$$\ln(1.2) = 7k$$

$$\tfrac{1}{7}\ln(1.2) = k$$

$$\therefore N = 1000e^{\frac{1}{7}\ln(1.2)\cdot t}$$

when $t = 12$

$$N = 1000e^{1/7\ln(1.2)\cdot 12}$$

$$\boxed{\begin{array}{l} N \approx 1366.90798 \\ \approx 1367 \text{ people} \end{array}}$$

33. [Calculator] Population y grows according to the equation $\frac{dy}{dt} = ky$, where k is a constant and t is measured in years. If the population doubles every 10 years, then the value of k is

You could solve this, but the solution is $y = y_0 e^{kt}$

when $t = 10$, $y = 2y_0$

$$2\cancel{y_0} = \cancel{y_0} e^{k(10)}$$

$$2 = e^{10k}$$

$$\ln(2) = 10K$$

$$\boxed{\tfrac{1}{10}\ln(2) = K}$$

34. [No Calculator] Bacteria in a certain culture increase at rate proportional to the number present. If the number of bacteria doubles in three hours, in how many hours will the number of bacteria triple?

Let B = # of Bacteria present

$$\frac{dB}{dt} = k \cdot B$$

$$\Rightarrow \quad B = B_0 e^{kt}$$

if $t = 3$, $B = 2B_0$

$$2\cancel{B_0} = \cancel{B_0} e^{k(3)}$$

$$2 = e^{3k}$$

$$\ln(2) = 3K$$

$$\tfrac{1}{3}\ln(2) = K$$

$$\therefore B = B_0 e^{\frac{1}{3}\ln(2) \cdot t}$$

What is t when $B = 3B_0$?

$$3\cancel{B_0} = \cancel{B_0} e^{\frac{1}{3}\ln(2) \cdot t}$$

$$3 = e^{\frac{1}{3}\ln(2) \cdot t}$$

$$\ln(3) = \tfrac{1}{3}\ln(2) \cdot t$$

$$\frac{3\ln 3}{\ln 2} = t$$

$$\boxed{\therefore t \approx 4.755 \text{ hours}}$$

35. Solutions to the differential equation $\frac{dy}{dx} = xy^3$ will also satisfy $\frac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

POINT $(1,2)$

$$\text{Slope @ } (1,2) = \left.\frac{dy}{dx}\right|_{(1,2)} = 1(2)^3 = 8$$

$$\therefore \boxed{y - 2 = 8(x-1)}$$

b) Use the tangent line equation from part *a* to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

$$f(1.1) \approx 8(1.1 - 1) + 2$$

$$= 2.8$$

Using a tangent line to approximate $f(1.1)$ will over approximate $f(1.1)$ if $f(x)$ is concave up & under approximate $f(1.1)$ if $f(x)$ is concave down.

On the interval $(1, 1.1)$ $\frac{d^2y}{dx^2} = y^3(1 + 3x^2y^2)$, $\frac{d^2y}{dx^2} > 0$ because all the x's are positive & we are told $f(x) > 0$ in that interval so all the y's are positive.

Since $\frac{d^2y}{dx^2} > 0$, $f(x)$ is concave up & the tangent line will OVER APPROXIMATE the function.

c) Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.

$$\int \frac{dy}{y^3} = \int x\,dx$$

NOTE: $y \neq 0$ so y is only defined on $(-\infty, 0)$ or $(0, \infty)$ since $y = 2$ is given y is on $(0, \infty)$

$$\int y^{-3} dx = \int x\,dx$$

$$-\tfrac{1}{2}y^{-2} = \tfrac{1}{2}x^2 + C$$

multiply the entire equation by -2 … Remember $-2 \cdot C = C$

$$y^{-2} = -x^2 + C$$

$$\frac{1}{y^2} = -x^2 + C$$

$$y^2 = \frac{1}{-x^2 + C}$$

when $x = 1$, $y = 2$ so,

$$2^2 = \frac{1}{-(1)^2 + C}$$

$$4 = \frac{1}{-1 + C}$$

$$-4 + 4C = 1$$

$$4C = 5$$

$$C = 5/4$$

$$\therefore y^2 = \frac{1}{-x^2 + 5/4}$$

$$\Rightarrow \boxed{y = \sqrt{\frac{1}{-x^2 + 5/4}}}$$

since $y > 0$