Calculus AB
Chapter 3.6 – 4.4 Review

KEY

CONCEPTS:

1. When looking for absolute extrema, where do the possible extrema exist, and how do you find them?

① Look for where the derivative equals zero or is undefined

② Look at endpoints ③ create a table for a "candidates test"

2. How do you justify relative extrema?

① "candidates test" (see #1) ③ 2ND Derivative Test

② Relative extrema can be justified by finding where the first derivative changes signs

Rel max: f' changes from (pos) to (neg)
Rel min: f' changes from (neg) to (pos)

3. How do you justify that a function is increasing or decreasing?

f is increasing if $f' > 0$
f is decreasing if $f' < 0$

4. How do you justify that a function is concave up or concave down?

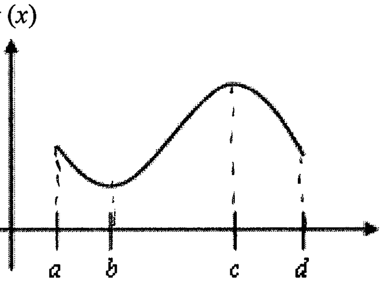
f is concave up if $f'' > 0$
f is concave down if $f'' < 0$

5. How do you justify that a function has a point of inflection?

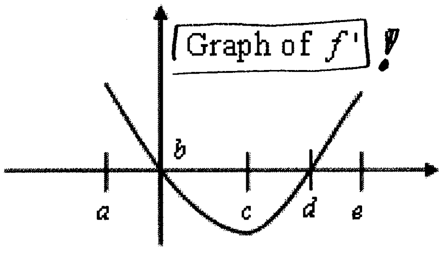
f has a point of inflection if f'' changes signs at that point.

6. Using the graph of $g(x)$ below, determine the signs of $g'(x)$ and $g''(x)$ at each point. Explain your reasoning.

At $x = a$ … $g'(x) < 0$ b/c g is dec / $g''(x) > 0$ b/c g is cu
At $x = b$ … $g'(x) = 0$ [Horizontal Tangent Line] / $g''(x) > 0$ b/c g is cu
At $x = c$ … $g'(x) = 0$ [Horiz. Tang. Line] / $g''(x) < 0$ b/c g is CD
At $x = d$ … $g'(x) < 0$ b/c g is dec / $g''(x) < 0$ b/c g is CD



7. Given the graph of f' below answer each of the following questions, and justify your response with a statement that contains the phrase "since f' ________________ …"



a) When is f increasing?

f is increasing on $(a,b) \cup (d,e)$
since $f' > 0$

b) When is f decreasing?

f is decreasing on (b,d)
since $f' < 0$

c) When is f concave up?

f is concave up on (c,e)
since f' is increasing [$f'' > 0$]

d) When is f concave down?

f is concave down on (a,c)
since f' is decreasing [$f'' < 0$]

e) When does f have a relative maximum?

f has a rel. max. at $x = b$
since f' changes signs from (pos) to (neg) at $x = b$.

f) When does f have a relative minimum?

f has a rel. min. at $x = d$ since f' changes signs from (neg) to (pos) at $x = d$.

g) When does f have a point of inflection?

f has a point of inflection at $x = c$
since f' changes from decreasing to increasing [f'' changed signs]

SKILLS:

8. [Calculator Allowed] If $f(x)$ has an inverse, then $f\left(f^{-1}(x)\right)=x$. Find $\left(f^{-1}\right)'(2)$ if $f(x)=x^3+2x-1$.

$$(f^{-1})'(2)=\frac{1}{f'(f^{-1}(2))}=\frac{1}{f'(1)}=\frac{1}{3(1)^2+2}=\boxed{\frac{1}{5}}$$

First, find $f^{-1}(2)$ by setting $f(x)=2$.

$x^3+2x-1=2$

$x=1$ (on calculator)

Next, find $f'(x)$: $f'(x)=3x^2+2$

9. Find the value of c guaranteed by the MVT for $f(x)=4x^2+5x$ on the interval [–2, 1].

$f'(x)=8x+5$ $\quad f(-2)=4(-2)^2+5(-2)=16-10=6$

$f(1)=4(1)^2+5(1)=4+5=9$

MVT guarantees a value of c between -2 & 1 such that

$f'(c)=\dfrac{f(-2)-f(1)}{-2-1}$

$8c+5=\dfrac{6-9}{-3}$

$8c+5=1$

$8c=-4$

$\boxed{c=-1/2}$

10. [Calculator Allowed] Find the value of c guaranteed by the MVT for $f(x)=\sin x$ on the interval [4, 5].
[♫: For those of you doing this problem algebraically, the answer is NOT $c\approx 1.774\ldots$ Why?]

$f'(x)=\cos x$ $\quad f(4)=\sin(4)$

$f(5)=\sin(5)$

MVT: $f'(c)=\dfrac{f(5)-f(4)}{5-4}$

$\cos c=\sin(5)-\sin(4)$

solve on calculator using graph ...

$\boxed{c\approx 4.509}$

11. Find the following derivatives:

a) $y=\sin^{-1}\left(3x^2\right)$

$\boxed{y'=\dfrac{6x}{\sqrt{1-9x^4}}}$

b) $y=\tan^{-1}(\sin x)$

$\boxed{y'=\dfrac{\cos x}{1+\sin^2 x}}$

c) $y=\sec^{-1}\left(\frac{1}{x}\right)=\sec^{-1}(x^{-1})$

$\boxed{y'=\dfrac{-\frac{1}{x^2}}{\left|\frac{1}{x}\right|\sqrt{\frac{1}{x^2}-1}}}$

d) $y=5^{x^{-3}-8}$

$\boxed{y'=5^{x^{-3}-8}\cdot(\ln 5)\cdot\left[-3x^{-4}\right]}$

e) $y=e^{8x}$

$\boxed{y'=e^{8x}\cdot 8}$

f) $y=\log_4\left(\sqrt{9x^3-2}\right)$

$\boxed{y'=\dfrac{\frac{1}{2}(9x^3-2)^{-1/2}\cdot 27x^2}{\sqrt{9x^3-2}}\cdot\dfrac{1}{\ln 4}}$

g) $y=\ln\left(7x^2+3\right)$

$\boxed{y'=\dfrac{14x}{7x^2+3}}$

h) $y=3^{\sec(x)}$

$\boxed{y'=3^{\sec(x)}\cdot(\ln 3)\cdot(\sec x\tan x)}$

i) $y=e^{\ln x}=x$ ← EASY WAY

$\boxed{y'=1}$

OR...

$y'=\underbrace{e^{\ln x}\cdot\frac{1}{x}}_{\text{simplifies to } y'=1}$

While none of the previous derivative questions included product and quotient rules, you should be able to combine these rules with any rules we have learned before. See your quiz from 3.8 and 3.9 for examples.

12. Suppose that functions f and g and their first derivatives have the following values at $x = -1$ and $x = 0$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	0	-1	2	1
0	-1	-3	-2	4

Find the first derivative of the following combinations at the given value of x.

a) $f(g(x))$ at $x=-1$ $\Rightarrow f'(g(-1))\cdot g'(-1)$

$= f'(-1)\cdot g'(-1)$

$= 2 \cdot 1$

$= \boxed{2}$

b) $\boxed{f^2(x)g^3(x)}$ at $x=0$ $\rightarrow [f(x)]^2\cdot[g(x)]^3$

$[f(0)]^2 \cdot 3[g(0)]^2 \cdot g'(0) + [g(0)]^3 \cdot 2[f(0)]\cdot f'(0)$

$= (-1)^2 \cdot 3(-3)^2\cdot(4) + (-3)^3 \cdot 2(-1)\cdot(-2) = \boxed{0}$

c) $g(f(x))$ at $x=-1$

$= g'(f(-1))\cdot f'(-1)$

$= g'(0)\cdot f'(-1) = 4 \cdot 2 = \boxed{8}$

d) $g(x+f(x))$ at $x=0$

$= g'(0+f(0))\cdot[1+f'(0)]$

$= g'(0+-1)\cdot[1+f'(0)] =$

$= g'(-1)\cdot[1+f'(0)]$

$= 1\cdot[1+-2]$

$= \boxed{-1}$

13. Find $\frac{dy}{dx}$ if $x^2y+3y^2=x-2$

$\left[x^2\frac{dy}{dx}+y\cdot 2x\right]+6y\,dy/dx = 1$

$x^2\,dy/dx + 6y\,dy/dx = 1-2xy$

$dy/dx\,(x^2+6y) = 1-2xy$

$\boxed{dy/dx = \frac{1-2xy}{x^2+6y}}$

14. Find $y'''(x)$ if $y=(4x+1)^{10}$

$y' = 10(4x+1)^9\cdot 4 = 40(4x+1)^9$

$y'' = 360(4x+1)^8\cdot 4 = 1440(4x+1)^8$

$y''' = 11520(4x+1)^7\cdot 4 = \boxed{80640(4x+1)^7}$

15. Suppose $y=x^3-3x$. [No Calculator]

a) Find the zeros of the function. $x^3-3x=0$

$x(x^2-3)=0$

$\boxed{x=0 \text{ or } x=\pm\sqrt{3}}$

b) Determine where y is increasing or decreasing and justify your response.

$y' = 3x^2-3 = 3(x^2-1) = 3(x+1)(x-1)$

y' pos | -1 | neg | 1 | pos

y is increasing on $(-\infty,-1]\cup[1,\infty)$ since $y'>0$

y is decreasing on $[-1,1]$ since $y'<0$

c) ~~Determine all~~ local extrema and justify your response.

There is a rel. max at $x=-1$ since y' changed signs from + to −. $\boxed{\text{Rel max} = 2}$

There is a rel. min at $x=1$ since y' changed signs from − to + $\boxed{\text{Rel min} = -2}$

d) Determine the points where y is concave up or concave down, and find any points of inflection. Justify your responses.

$y''=6x$

y'' neg | 0 | pos

y is concave up on $(0,\infty)$ since $y''>0$

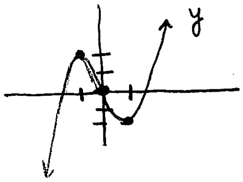
y is concave down on $(-\infty,0)$ since $y''<0$

There is a point of inflection on y when $x=0$ since y'' changed signs at $x=0$.

$\boxed{\text{P.O.I.} = (0,0)}$

e) Use all your information to sketch a graph of this function.

Known Points: $(-1,2)$
$(1,-2)$
$(0,0)$



y'	pos		neg		neg		pos
		-1		0		1	
y''	neg		neg		pos		pos
y	INC CD		DEC CD		DEC CU		INC CU

16. If $f'(x)=x^2-9x+1$, what does $f(x)$ equal?

Given $f'(x)$

Go "Backwards"

$$\boxed{f(x)=\frac{1}{3}x^3-\frac{9}{2}x^2+x+C}$$

Don't forget the "C"

17. Suppose the acceleration of an object in terms of time is given by $a(t) = 5$.

Derivatives

P

V

A

UNDO Derivatives

a) What is the velocity function if $v(2) = 10$?

$v(t) = 5t + C$

if $v = 10$ when $t = 2$

$10 = 5(2) + C$

$0 = C$

$$\boxed{\therefore v(t) = 5t}$$

b) Using your velocity function from part *a*, what is the position function if $s(0) = 5$?

$s(t) = \frac{5}{2}t^2 + C$

if $s = 5$ when $t = 0$, $5 = \frac{5}{2}(0)^2 + c$

$5 = C$

$$\boxed{\therefore s(t) = \frac{5}{2}t^2 + 5}$$

18. Suppose $\boxed{\frac{d^2y}{dx^2}=x^3-4x^2}$. Justify each response below.

$y'' = x^2(x-4)$

y'': neg | 0 | neg | 4 | pos

a) Where is *y* concave up?

y is concave up on $(4,\infty)$ since $y''>0$

b) Where is *y* concave down?

y is concave down on $(-\infty,0)\cup(0,4)$ since $y''<0$

c) Are there any inflection points on *y*? If so, where?

Since y'' changes signs at $x=4$, there is a point of inflection when $x=4$.

THERE IS NO POI @ $x=0$ b/c y'' DID NOT CHANGE SIGNS at $x=0$.

SKILLS AND CONCEPTS APPLIED

19. [Calculator Allowed] The derivative of $h(x)$ is given by $h'(x)=2\cos\left(x-\frac{\pi}{6}\right)+1$ on the interval $[-2\pi,2\pi]$. Justify EVERY response.

(graphed below]

a) Where is *h* (*x*) increasing?

h is increasing when $h'(x)>0$

This occurs on $[-2\pi,-3.665]\cup[-1.571,2.618]\cup[4.712,2\pi]$

b) Where is *h* (*x*) concave down?

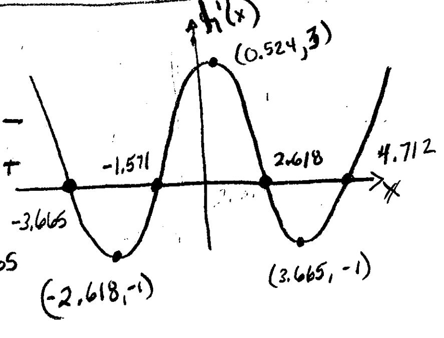
h is concave down when h' is decreasing ($h''<0$)

This occurs on $(-2\pi,-2.618)\cup(.524,3.665)$

c) Find ~~all~~ x-coordinates of all extrema of *h* (*x*) on the interval $[-2\pi,2\pi]$.

Rel max on h(x) at $x=-3.665$ & $x=2.618$ since h' changes from + to −

Rel min on h(x) at $x=-1.571$ & $x=4.712$ since h' changes from − to +



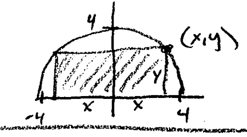
d) Does *h* (*x*) have a point(s) of inflection? If so, where?

h has a point of inflection at $x=-2.618$, $x=0.524$, & $x=3.665$

h' changes from decreasing to increasing (or vice-versa)

★★ IF YOU DON'T USE CALCULUS FOR #20 & 21 YOU WON'T GET CREDIT ON THE TEST!

20. Find the maximum area of a rectangle inscribed under the curve $f(x)=\sqrt{16-x^2}$.



$A = 2xy = 2x\sqrt{16-x^2}$ Domain: $[0,4]$

$A' = 2x \cdot [\tfrac{1}{2}(16-x^2)^{-1/2} \cdot (-2x)] + \sqrt{16-x^2} \cdot [2]$

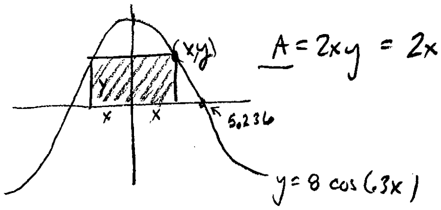
$\boxed{A'} = \dfrac{-2x^2}{\sqrt{16-x^2}} + 2\sqrt{16-x^2} = \dfrac{-2x^2+2(16-x^2)}{\sqrt{16-x^2}} = \dfrac{-2x^2+32-2x^2}{\sqrt{16-x^2}} = \boxed{\dfrac{32-4x^2}{\sqrt{16-x^2}}}$

$A' = 0$ when
$32-4x^2=0$
$\pm 2\sqrt{2} = x$

x	0	$2\sqrt{2}$	4
A	0	16	0

MAX AREA = 16

21. [Calculator Allowed] A rectangle is inscribed under one arch of $y=8\cos(0.3x)$ with its base on the x-axis and its upper two vertices on the curve symmetric about the y-axis. What is the largest area the rectangle can have?



$A = 2xy = 2x \cdot 8\cos(.3x) = 16x\cos(.3x)$ Domain: $[0, 5.236]$

$A' = 16x \cdot [-\sin(.3x) \cdot (.3)] + \cos(.3x) \cdot [16]$

$A' = 0$ at $x \approx 2.8677786...$
(STORE AS B)

x	A
0	0
B	29.925
5.236	0

MAX AREA ≈ 29.925

22. The function f is continuous on [0, 3] and satisfies the following:

x	0	$0<x<1$	1	$1<x<2$	2	$2<x<3$	3
f	0	Neg	–2	Neg	0	Pos	3
f'	–3	Neg	0	Pos	DNE	Pos	4
f''	0	Pos	1	Pos	DNE	Pos	0

a) Find the absolute extrema of f and where they occur.

Check endpoints & CRITICAL POINTS

$x=0$
$x=3$

$f'=0$	f' DNE
$x=1$	$x=2$

x	0	1	2	3
f(x)	0	-2	0	3

MAX = 3
MIN = -2

b) Find any points of inflection.

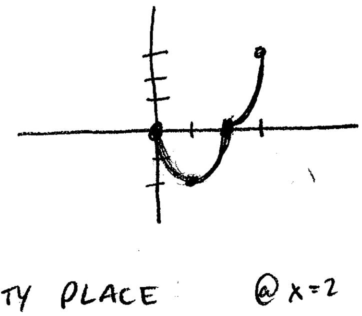
where f'' changes signs ... NONE, f'' is always POSITIVE

c) Sketch a possible graph of f.

KNOWN POINTS
(0,0)
(1,-2)
(2,0)
(3,3)

f'	−	+	+
f''	+	+	+
f	DEC CU	INC CU	INC CU

0 1 2 3



NOTE... POINTY PLACE @ x=2 b/c f'(2) DNE

Go back and Review the questions from your assignments in this chapter … especially those in section 4.3.

Understand your notecards!