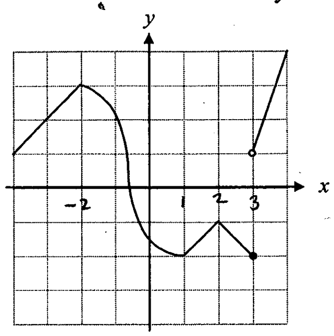
Calculus
Chapter 3 Review Sheet

1. The function for $f(x)$ is graphed below.
There is a vertical tangent line when $x = -\frac{1}{2}$
Where is $f(x)$ NOT differentiable? Why?



at $x = -2$, $x = 1$, $x = 2$ } Pointy Places

at $x = 3$ Discontinuity

at $x = -\frac{1}{2}$ Vertical Tangent Line

2. If $f(x)$ has a derivative at $x = 2$, tell whether or not each of the following must be true?

a) $\lim_{x \to 2} f(x)$ exists

b) $f'(2)$ exists

c) $f''(2)$ exists. Doesn't have to be True

d) $f(x)$ is continuous at $x = 2$.

e) $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$ exists.

f) $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$ exists.

if f has a derivative at $x=2$,

$f'(2)$ exists (B)

$\Downarrow$

f(x) is continuous at $x=2$ (D)

$\Downarrow$

$\lim_{x \to 2} f(x) = f(2)$ (A)

(E) & (F) are definitions of a derivative

3. Use the $\boxed{\text{alternative definition of the derivative}}$ $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ to find each of the following:

a) $f'(x)$ if $f(x) = \frac{3}{x}$.

$$\lim_{x \to a} \left(\frac{\frac{3}{x} - \frac{3}{a}}{x - a} \right) = \lim_{x \to a} \left(\frac{3a - 3x}{ax} \cdot \frac{1}{x - a} \right)$$

$$= \lim_{x \to a} \left(\frac{-3(x-a)}{ax} \cdot \frac{1}{x-a} \right) = \lim_{x \to a} \frac{-3}{ax}$$

$$= \frac{-3}{a \cdot a}$$

$$= \frac{-3}{a^2}$$

$$\boxed{\therefore f'(x) = -\frac{3}{x^2}}$$

b) $\frac{dy}{dx}$ if $f(x) = 34$

$$\lim_{x \to a} \left(\frac{34 - 34}{x - a} \right) = \lim_{x \to a} \left(\frac{0}{x - a} \right)$$

$$= \lim_{x \to a} (0) = 0$$

$$\boxed{\frac{dy}{dx} = 0}$$

c) $y'(1)$ if $y = 3x^2 + 5x$

$$y'(a) = \lim_{x \to a} \left(\frac{(3x^2 + 5x) - (3a^2 + 5a)}{x - a} \right)$$

$$= \lim_{x \to a} \frac{3x^2 - 3a^2 + 5x - 5a}{x - a}$$

$$= \lim_{x \to a} \frac{3(x^2 - a^2) + 5(x - a)}{x - a}$$

$$= \lim_{x \to a} \frac{3(x+a)(x-a) + 5(x-a)}{x - a}$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)}\,[3(x+a) + 5]}{\cancel{x-a}}$$

$$= 3(a+a) + 5$$

$$= 6a + 5$$

$$\therefore y'(1) = 6(1) + 5 = \boxed{11}$$

4. Find the following limits: (if you are spending a lot of time on this, you aren't "seeing" the point)

a) $\lim_{h \to 0} \frac{\tan\left(\frac{\pi}{4} + h\right) - \tan\left(\frac{\pi}{4}\right)}{h}$

This is $y'(\pi/4)$ for $y = \tan x$

$y' = \sec^2 x$, so $y'(\pi/4) = \sec^2\left(\frac{\pi}{4}\right) = \boxed{2}$

b) $\lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} = \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h}$

This is $y'(4)$ for $y = \sqrt{x}$

$y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$, so $y'(4) = \frac{1}{2\sqrt{4}} = \boxed{\frac{1}{4}}$

5. If $f(x) = \begin{cases} 2ax^2 + b & x \ge 1 \\ -3x + 4 & x < 1 \end{cases}$, find a and b so that f is both continuous and differentiable.
(Be sure to use definitions to justify your work)

$$f'(x) = \begin{cases} 4ax & \text{if } x > 1 \\ -3 & \text{if } x < 1 \end{cases}$$

To be continuous at $x = 1$,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$-3(1) + 4 = 2a(1)^2 + b = 2a(1)^2 + b$$

$$\boxed{1 = 2a + b}$$

To be differentiable at $x = 1$,

$$f'(x) \text{ as } x \to 1^- = f'(x) \text{ as } x \to 1^+$$

$$-3 = 4a(1)$$

$$-3 = 4a$$

$$\boxed{-\frac{3}{4} = a}$$

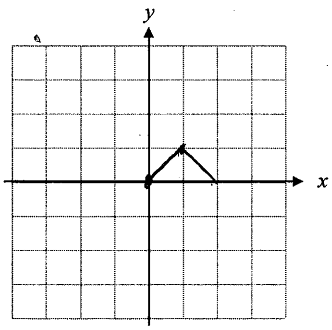
$$\therefore 1 = 2(-3/4) + b$$

$$1 = -\frac{3}{2} + b$$

$$\boxed{\frac{5}{2} = b}$$

6. Use the function $f(x)=\begin{cases} x & \text{if } 0\le x\le 1 \\ 2-x & \text{if } 1<x\le 2 \end{cases}$.

a) Graph the function



b) Is f continuous at $x = 1$? Explain.

Yes, $\underbrace{\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)}_{\text{All equal 1}}$

c) Is f differentiable at $x = 1$? Explain.

No, there is a "pointy place"

"aka" the Left & Right Derivatives are not equal

7. Suppose $f(x)=\begin{cases} 2x-3 & \text{if } -1\le x<0 \\ x-3 & \text{if } 0\le x\le 4 \end{cases}$

$f'(x) = \begin{cases} 2 & \text{if } -1<x<0 \\ 1 & \text{if } 0<x<4 \end{cases}$

a) Where is the function differentiable?

$(-1,0) \cup (0,4)$ ... NOT differentiable at $x=0$ b/c Left & Right Derivatives are different

b) Where is the function continuous but not differentiable?

Since $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$, $f(x)$ is continuous $\boxed{\text{at } x=0}$

(but we already said is was not diff.)

$\lim_{x\to 0^-} f(x) = 2(0)-3 = -3$

$\lim_{x\to 0^+} f(0) = 0-3 = -3$

$f(0) = 0-3 = -3$

c) Where is the function neither continuous nor differentiable?

Nowhere on the Domain

8. Find the equation of the *tangent line* to the curve $y = 2\underbrace{\sin x}_{u}\underbrace{\cos x}_{v}$ at $x=\frac{\pi}{2}$.

Point $(\pi/2, 0)$

$2\sin(\pi/2)\cos(\pi/2) = 2(1)(0)$

PRODUCT RULE

$y' = 2\sin x[-\sin x] + \cos x[2\cos x]$

$y' = -2\sin^2 x + 2\cos^2 x$

Slope $= y'(\pi/2) = -2[\sin(\pi/2)]^2 + 2[\cos(\tfrac{\pi}{2})]^2$

$= -2(1)^2 + 2(0)^2 = -2$

$\boxed{y - 0 = -2(x - \pi/2)}$

9. Find the derivative of each of the following:

a) $y=x^{-5}-\dfrac{x^3}{8}+\dfrac{1}{4}\sqrt[3]{x}$

Rewrite 1st: $y = x^{-5} - \frac{1}{8}x^3 + \frac{1}{4}x^{1/3}$

$\boxed{y' = -5x^{-6} - \frac{3}{8}x^2 + \frac{1}{12}x^{-2/3}}$

b) $y = 3\underbrace{\sec x}_{u}\underbrace{\csc x}_{v}$

PRODUCT RULE!

$\boxed{y' = 3\sec x[-\csc x\cot x] + \csc x(3\sec x\tan x)}$

$= \dfrac{-3}{\sin^2 x} + \dfrac{3}{\cos^2 x}$

$= -3\csc^2 x + 3\sec^2 x$

c) $y=\dfrac{2x+1}{3x-4}$ QUOTIENT RULE!

$\boxed{y' = \dfrac{(3x-4)(2) - (2x+1)(3)}{(3x-4)^2}}$

$y' = \dfrac{6x-8-6x-3}{(3x-4)^2}$

$\boxed{y' = \dfrac{-11}{(3x-4)^2}}$

10. Using values from the chart, estimate a value for $f'(3)$ and explain its meaning in the context of this problem. Show how you arrived at your answer.

Slope at $x=3$ use Points just to left & right at $x=3$

x = minutes	$f(x)$ = \$
1	4
2	6
3	9
4	11

$f'(3) \approx \dfrac{11-6}{4-2} = \dfrac{5}{2}$

f is increasing \$5/2 per minute at $x=3$

11. [No Calculator] Suppose $x(t)=t^2-8t+12$ is a position of a particle moving along the x axis at time t.

$x(3)=(3)^2-8(3)+12=$
$=9-24+12$
$=-3$

a) Find the average velocity for the first 3 seconds.

average rate of change in position $\quad \dfrac{x(3)-x(0)}{3-0}=\dfrac{-3-12}{3}=-5$

b) Find the velocity at $t=4$ seconds.

$v(t)=x'(t)=2t-8 \qquad v(4)=2(4)-8=0$

c) When is the object stopped?

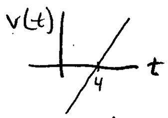
when $v(t)=0$ ... this occurs at $t=4$

d) When is the acceleration of the object 0?

$a(t)=v'(t)=2 \qquad$ since $a(t)=2$, $a(t)\neq 0$.

e) When does the object change direction?

when $v(t)$ changes signs.



This occurs at $t=4$

f) When does the object slow down?

when $v(t)$ & $a(t)$ have different signs

Since $a(t)=2$, $a(t)>0$ all the time, so we need to know when $v(t)<0$ ... this occurs when $t<4$

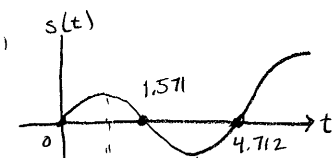
g) When is the object moving left?

When $v(t)<0$ ... this occurs when $t<4$

12. [Calculator] A particle is moving along the x-axis and its position function at time t is given by the equation $s(t)=\sqrt{t}\cos t$, where $0\le t\le 2\pi$.

a) Find the zeros of $s(t)$. What does this tell you? $\quad t\approx 1.571$ $(\pi/2)$, $t\approx 4.712$ $(3\pi/2)$

when the particle's position is at position 0

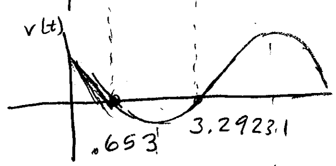


b) Find the velocity of the object at any time t.

$v(t)=s'(t)=\sqrt{t}\,[-\sin t]+\cos t\,[\tfrac{1}{2}\cdot t^{-1/2}]$

c) Find the zeros of $v(t)$. What does this tell you? $t\approx .653$, $t\approx 3.292$

when $v(t)=0$, the particle is stopped

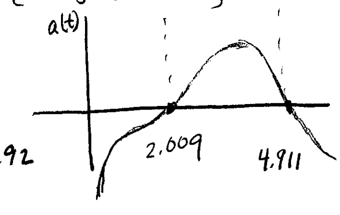


d) Find the acceleration of the object at any time t.

$a(t)=v'(t)=\left(\sqrt{t}\,[-\cos t]+(-\sin t)\,[\tfrac{1}{2}t^{-1/2}]\right)+\left(\cos t\,[-\tfrac{1}{4}t^{-3/2}]+[\tfrac{1}{2}t^{-1/2}]\cdot(-\sin t)\right)$

e) Find the zeros of $a(t)$.

$a(t)=0$ when $t\approx 2.009$ & $t\approx 4.911$



d) When does the object change direction? Justify your response.

Object changes direction when $v(t)$ changes signs ... @ $t\approx .653$ & $t\approx 3.292$

f) When does the object speed up? When does it slow down? Justify your response.

SPEED UP: $a(t)$ & $v(t)$ have same signs $\quad (.653, 2.009)\cup(3.292, 4.911)$

SLOW DOWN: when $a(t)$ & $v(t)$ have different signs $\quad (0, .653)\cup(2.009, 3.292)\cup(4.911, 2\pi)$

v(t): + − − + +
a(t): − − + + −
0, .653, 2.009, 3.292, 4.911, 2π

13. On Earth, if you shoot a small object 64 feet straight up in the air, the object will be $h(t)=64t-16t^2$ feet above your head at t seconds after launching.

a) Find $\dfrac{dh}{dt}$ and $\dfrac{d^2h}{dt^2}$ and explain what you have found.

$dh/dt=$ velocity of object at time t in ft/sec

$dh/dt=64-32t \quad$ & $\quad \dfrac{d^2h}{dt^2}=-32$

$d^2h/dt^2=$ acceleration ... a constant -32 ft/sec²

b) Is the object speeding up or slowing down at $t=1$ second? Justify your response.

at $t=1$

$v(1)=64-32(1)=32$ & $a(1)=-32$

Since $a(1)$ & $v(1)$ have different signs, the object is slowing down.

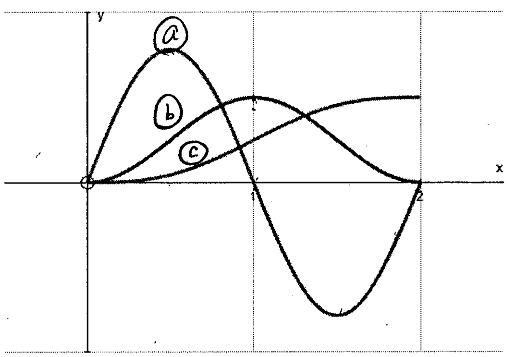
14. If a hemispherical bowl of radius 10 in. is filled with water to a depth of x in., the volume of water is given by $V = \pi\left[10 - \frac{x}{3}\right]x^2$.

Find the rate of increase of the volume. Indicate units of measure. since Volume is measured in in^3 & x is measured in inches,

$\boxed{V'(x) = \pi\left[10 - \frac{x}{3}\right] \cdot 2x + x^2 \cdot \pi\left[-\frac{1}{3}\right]} = 20\pi x - \frac{2\pi}{3}x^2 - \frac{\pi}{3}x^2 = 20\pi x - \pi x^2$

$\frac{dV}{dt} = \frac{in^3}{in} \Downarrow \boxed{in^2}$

15. The following graphs show the distance traveled, velocity, and the acceleration for each second of a 2-minute automobile trip. Which graph shows distance? Velocity? Acceleration? Explain your choice.



We Know that

$P' = V$

and

$P'' = V' = A$

Look for where derivatives = 0

$a = b'$

a' is NOT on the graph

POSITION = c

VELOCITY = b

ACCELERATION = a

$c' > 0$ for all values of t

16. Suppose that a function f and its first derivative have the following values at $x = 0$ and $x = 1$.

Find an expression for the derivative of the following combinations, then find the derivative at the indicated point.

x	$f(x)$	$f'(x)$
0	9	–2
1	–3	1/5

a) $\sqrt{x}f(x)$ at $x = 1$

Derivative $= \sqrt{x} \cdot f'(x) + f(x) \cdot \frac{1}{2}x^{-1/2}$

Derivative at $x=1$ $\Big\} = \sqrt{1} \cdot f'(1) + f(1) \cdot \frac{1}{2}(1)^{-1/2}$

$= 1 \cdot \frac{1}{5} + -3 \cdot \frac{1}{2} \cdot 1$

$= \frac{1}{5} - \frac{3}{2} = \frac{2-15}{10} = \boxed{-13/10}$

b) $\dfrac{f(x)}{2+\cos x}$ at $x = 0$

Derivative $= \dfrac{[2+\cos x] \cdot f'(x) - f(x) \cdot [-\sin x]}{[2+\cos x]^2}$

Derivative at $x = 0$ $\Big\} = \dfrac{[2+\cos(0)] \cdot f'(0) - f(0) \cdot [-\sin(0)]}{[2+\cos(0)]^2} = \dfrac{(2+1)\cdot(-2) - 9(-0)}{(2+1)^2}$

$= \frac{-6}{9} = \boxed{-2/3}$

17. Suppose that functions f and g and their first derivatives have the following values at $x = -1$ and $x = 0$.

Find an expression for the derivative of the following combinations , then find the derivative at the indicated point.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
–1	0	–1	2	1
0	–1	–3	–2	4

a) $3f(x) - g(x)$ at $x = -1$

Derivative $= 3f'(x) - g'(x)$

Derivative at $x=-1$ $\Big\} = 3 \cdot f'(-1) - g'(-1)$

$= 3(2) - (1)$

$= \boxed{5}$

b) $\dfrac{f(x)}{g(x)+2}$ at $x = 0$

Derivative $= \dfrac{[g(x)+2] \cdot f'(x) - f(x) \cdot [g'(x)]}{[g(x)+2]^2}$

Derivative at $x = 0$ $\Big\} = \dfrac{[g(0)+2] \cdot f'(0) - f(0) \cdot g'(0)}{[g(0)+2]^2}$

$= \dfrac{[-3+2]\cdot(-2) - (-1)\cdot(4)}{[(-3)+2]^2} = \dfrac{2+4}{1} = \boxed{6}$

c) $f(x) \cdot 4g(x)$ at $x = 0$

Derivative $= f(x) \cdot 4g'(x) + 4g(x) \cdot f'(x)$

Derivative at $x=0$ $\Big\} = f(0) \cdot 4g'(0) + 4g(0) \cdot f'(0)$

$= (-1) \cdot 4(4) + 4(-3) \cdot (-2)$

$= -16 + 24$

$= \boxed{8}$

18. When finding marginal profit (or marginal cost, or marginal revenue) you find the derivative and plug in the current level of production ?

19. If $P(x) = 4x^3 - 7x - 10$ is the equation for profit on x items, find the marginal profit of the 12^{th} item. (currently producing 11)

$P'(\cancel{x}) = 12x^2 - 7$ $\quad P'(11) = 12(11)^2 - 7 = \boxed{\$1445}$

20. Use your calculator to find the derivative of the following functions at the indicated point. Label each correctly.

a) $y = \sqrt{3x-8}$ when $x = 12$ $\quad \boxed{y'(12) \approx .283}$

b) $f(x) = \cos(3x)$ when $x = \pi/2$ $\quad \boxed{f'(\pi/2) \approx 3.000}$

c) $P(x) = \dfrac{x^2 \sin x + \tan x}{x+7}$ when $x = 0$. $\quad \boxed{P'(0) \approx .143}$

21. [Calculator] The number of the zombies in a small town is modeled by the equation $Z(t) = \dfrac{800}{1+e^{-.05t}}$, where t is the number of days since the initial outbreak and $Z(t)$ is the total number of zombies.

a) Calculate the number of zombies initially. (t = 0)

$Z(0) = 400$

b) What is the rate of change in the number of zombies on day 5? Indicate units of measure.

$Z'(5) \approx 9.845$ $\quad$ zombies/day

the # of zombies is increasing 9.845 zombies/day

(and since they're zombies, it's ok to have "parts" of zombies) :)