6.2 INTEGRATION BY SUBSTITUTION

Notecards from Section 6.2: *U* – Substitution; Integrals of TRIG and TRIG^2 Functions; Algebraic Techniques of Integration

Up to this point you have been finding antiderivatives of functions that followed directly from their derivative rules. Today we learn how to undo a function that used the chain rule to find the derivative. First, a quick review of the Chain Rule …

$$\frac{d}{dx}\left(f\left(g\left(x\right)\right)\right)=f'\left(g\left(x\right)\right)\cdot g'\left(x\right).$$

Outside derivative — Inside derivative

The Chain Rule gives us a product of two factors: the "outside derivative" and the "inside derivative." So, if a function has this form then it has an antiderivative.

Example 1: Use the Chain Rule to differentiate $f(x)=\left(3x^4-5\right)^{12}$.

Example 2: Tell whether or not each antiderivative is going to undo a chain rule.

a) $\int\left(x^2-1\right)^3 2x\,dx$ b) $\int 3x^2\sqrt{x^3+2}\,dx$ c) $\int x\left(x^3-5\right)dx$

Substitution Method

Let u be the "inside function" and $du=u'\left(x\right)dx$

It is important to note, that with substitution, the goal is to substitute ALL values of the integrand with either *u* or *du*. Any "extras" must be accounted for, or substitution will NOT work!

Indefinite Integration with Substitution

Example 3: $\int x^3\sqrt{x^4+2}\,dx$

Example 4: $\int \sin^2\left(3x\right)\cos\left(3x\right)dx$

Example 5: $\int \frac{x}{x^2+2}\,dx$

Definite Integrals with Substitution

Indefinite Integrals needed a " +C " at the end of every antiderivative … Definite Integrals have limits.

If you change the variables, the limits still refer to the original variable … How will you decide to deal with those limits? … you have two choices …

#1: Leave the limits in terms of the original variable and integrate like you did for the indefinite integrals. Once you have returned all variables back to the original letter, you can plug in the upper limits and lower limits.

#2: Using the rule for the change of variables, change the limits with the same rule … then you never need to return to the original variable.

♬: THE LIMITS MUST MATCH THE VARIABLE BEING USED, OR THERE MUST BE SOME NOTATION TO INDICATE THAT THE LIMITS BEING USED ARE DIFFERENT FROM THE VARIABLE BEING USED!

Method 1: Use substitution to evaluate the integral, but do NOT change the upper and lower limits.

Example 6: Compute $\int_0^1 x\sqrt{1-x^2}\, dx$.

Method 2: Use substitution to evaluate the integral, and change the limits using the substitution rule you created.

Example 7: Evaluate $\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\, dx$.

Example 8: Compute $\int_0^{\pi/4} \frac{\tan^3 \theta}{\cos^2 \theta}\, d\theta$.

Algebraic Techniques

When substitution doesn't work, you sometimes just need to "massage" the problem into a form that will work using some algebraic techniques.

Long Division … when the numerator has a degree greater than or equal to the denominator

Example 9: $\int \frac{x^2 - 1}{x^2 + 1} \, dx$

Expand … when the "inside" doesn't have a derivative on the outside , try expanding the function

Example 10: $\int (\sin x + \cos x)^2 \, dx$

Complete the Square … useful when you have a x^2 and x term in the denominator but no x term in the numerator.

Example 11: $\int \frac{2 \, dx}{x^2 - 6x + 10}$

Separate the numerator … when you have more than one term in the numerator

Example 12: $\int \frac{3x + 2}{\sqrt{1 - x^2}} \, dx$

Integrals of Trigonometric and Squared Trigonometric Functions

First, a few identities from trigonometry that you may or may not remember.

1. $1+\tan^2 x=\sec^2 x$ … everyone remembers this one, right?!

2. $\cos(2x)=2\cos^2 x-1$ … which can be rewritten to $\cos^2 x=\dfrac{1+\cos(2x)}{2}$

3. $\cos(2x)=1-2\sin^2 x$ … which can be rewritten to $\sin^2 x=\dfrac{1-\cos(2x)}{2}$

Can you integrate all of these functions? The first 4 should already be known.

$\int \sin x\,dx$ $\int \cos x\,dx$ $\int \sec^2 x\,dx$ $\int \csc^2 x\,dx$

$\int \tan x\,dx$ $\int \cot x\,dx$

$\int \csc x\,dx$ $\int \sec x\,dx$

$\int \sin^2 x\,dx$ $\int \cos^2 x\,dx$

$\int \tan^2 x\,dx$ $\int \cot^2 x\,dx$

Examples with Inverse Trigonometric Functions

Example 13: $\int \frac{dx}{\sqrt{4-x^2}}$

Example 14: $\int \frac{dx}{x\sqrt{4x^2-9}}$