6.2 INTEGRATION BY SUBSTITUTION

Up to this point, you have only been integrating functions by recognizing the function as an antiderivative of an elementary function. Integration by substitution allows us to integrate a function that was obtained by using the chain rule to take the derivative of a composite function.

The Guess-and-Check Method

Recall the way we differentiate composite functions using the Chain Rule:

$$\frac{d}{dx}\left(f\left(g\left(x\right)\right)\right)=f'\left(g\left(x\right)\right)\cdot g'\left(x\right).$$

Outside derivative Inside derivative

The Chain Rule gives us a product of two factors: the "outside derivative" and the "inside derivative." So, if a function has this form then it has an antiderivative.

Example: Use the Chain Rule to differentiate the following functions and state the outside and inside derivative.

a) $f(x)=\left(3x^4-5\right)^{12}$

b) $g(x)=\sin\left(x^3\right)$

Example. Use the pattern above to find the antiderivative of the following functions.

a) $\int\left(x^2-1\right)^3 2x\,dx$

b) $\int 3x^2\sqrt{x^3+2}\,dx$

c) $\int x\left(x^3-5\right)dx$

What problem did you find with the part (c)? How did you correct this problem?

Substitution Method

When the integrand is complicated, it helps to formalize this guess – and – check method as follows:

To Make a Substitution

Let *u* be the "inside function" and $du=u'(x)\,dx$

It is important to note, that with substitution, the goal is to substitute ALL values of the integrand with either *u* or *du*. Any "extras" must be accounted for, or substitution will NOT work!

Indefinite Integration with Substitution

Example: $\int x^3\sqrt{x^4+2}\ dx$

Example: $\int \sin^2(3x)\cos(3x)\ dx$

Example: $\int \tan x\ dx$

Example: $\int (x+1)\sqrt{2-x}\ dx$

Example: $\int \frac{x}{x^2+2}\ dx$

Example: $\int \frac{2x}{\sqrt{x^2+6}}\ dx$

Example: $\int \frac{e^x}{e^x+4}\ dx$

Example: $\int \frac{e^x+4}{e^x}\ dx$

Definite Integrals with Substitution

The only difference between indefinite integrals with substitution and definite integrals with substitution is the way in which you treat the limits of integration. Once you change variables, the limits no longer apply to the new variable.

Two methods: The first method involves using substitution to change the variables, then changing BACK into the original variable as before *before* you evaluate the definite integral. The second method involves using substitution to change the variable *and* changing the limits to correspond to that new variable. You then evaluate the definite integral using the converted limits.

♬: THE LIMITS MUST MATCH THE VARIABLE BEING USED, OR THERE MUST BE SOME NOTATION TO INDICATE THAT THE LIMITS BEING USED ARE DIFFERENT FROM THE VARIABLE BEING USED!

Method 1: Compute the indefinite integral, expressing the antiderivative in terms of the original variable, and then evaluate the result at the original limits.

Example: Compute $\int_0^1 x\sqrt{1-x^2}\, dx$.

Method 2: Convert the original limits to new limits in terms of the new variable and do not convert the antiderivative to back to the original variable.

Example: Evaluate $\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\, dx$.

Example: Compute $\int_0^{\pi/4} \frac{\tan^3\theta}{\cos^2\theta}\, d\theta$.

Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x, and let $a > 0$.

1. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$
2. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$
3. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

Using these formulas involves "integrating by recognition". If you can recognize that the integral fits a particular format, then you know right away what the integral is.

Example: $\int \frac{dx}{\sqrt{4 - x^2}}$

Example: $\int \frac{dx}{2 + 9x^2}$

Example: $\int \frac{dx}{x\sqrt{4x^2 - 9}}$

Sometimes, integrals do not quite fit any of these formulas. Other options include substitution and *completing the square*.

Example: $\int \frac{dx}{\sqrt{e^{2x} - 1}}$

Example: $\int \frac{x + 2}{\sqrt{4 - x^2}}\,dx$

Example: $\int \frac{dx}{x^2 - 4x + 7}$

Notecards from Section 6.2: *U* - Substitution