6.1 ANTIDERIVATIVES AND SLOPE FIELDS

Notecards from Section 6.1: Slope Fields (Revisited); Indefinite Integrals

Indefinite Integrals

In the previous chapter we dealt with definite integrals. Definite integrals had limits of integration . Indefinite integrals do not.

The set of all antiderivatives of a function $f(x)$ in the **indefinite integral of f with respect to x** and is denoted

$$\int f(x)\,dx$$

Recall, that all antiderivatives differ by a constant, so if $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + C$, where C is the constant of integration. The following table gives a list of results you should already be familiar with.

Integral Formulas

1. Power Rule for $n \neq -1$: $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$
2. Rule for $n = -1$: $\int \dfrac{1}{x}\,dx = \ln|x| + C$
3. $\int e^{kx}\,dx = \dfrac{e^{kx}}{k} + C$
4. $\int \sin(kx)\,dx = \dfrac{-\cos(kx)}{k} + C$
5. $\int \cos(kx)\,dx = \dfrac{\sin(kx)}{k} + C$
6. $\int \sec^2(x)\,dx = \tan(x) + C$
7. $\int \csc^2(x)\,dx = -\cot(x) + C$
8. $\int \sec(x)\tan(x)\,dx = \sec(x) + C$
9. $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$
10. $\int \dfrac{dx}{x^2+1} = \tan^{-1}(x) + C$
11. $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C$
12. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C$ … where a is a constant

Example 1: Evaluate each integral.

a) $\int \left(-x^{-3} + \sqrt[3]{x} - 1 + e^{3x}\right) dx$

b) $\int (3\sin x - \sin 3x)\,dx$

Differential Equations

A **differential equation** is an equation containing a derivative. Just like in Algebra, when you want to solve an equation, you use an inverse operation. To "undo" a derivative we take an ____________________________________.
Recall, that a function can have many antiderivatives, all of which vary by a _________________________.

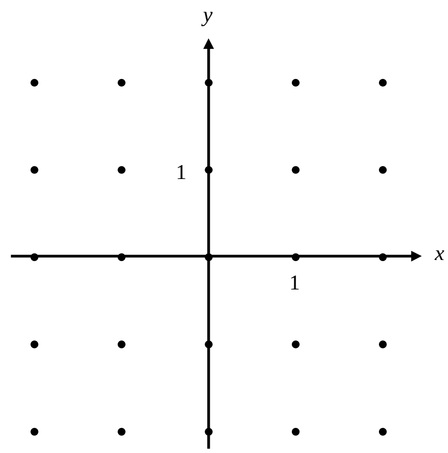
Solving a differential equation involves finding a unique equation that satisfies some *initial conditions* or *initial values*.
The **order** of a differential equation is the order of the highest derivative involved in the equation.

Example 2: Solve $\frac{dy}{dx} = \sin x$ by **separation of variables** if $y(0) = 2$.

A Graphical Look at Differential Equations

A **slope field** (or direction field) for the first order differential equation $\frac{dy}{dx} = f(x, y)$ is a plot of short line segments with slope $f(x, y)$ for a lattice of points (x, y) in the plane.

Example 3: On the diagram below, plot the slope field of the differential equation $\frac{dy}{dx} = 2y$.



Example 4: Suppose that you know that the point (0, –1) is on a particular solution of the differential equation above. By following slopes, draw on the diagrams above what you think the particular solution look like. (♬: The graph should follow the pattern of the slope field, but may go between the points rather than through them.)

Example 5: Solve the differential equation $\frac{dy}{dx} = 2y$ from the previous example by first **separating the variables**. Find the particular solution that contains the point given in the last example. Does your solution make sense when compared to the graph of the slope field?

6.2 INTEGRATION BY SUBSTITUTION

Notecards from Section 6.2: *U* – Substitution; Integrals of TRIG and TRIG^2 Functions; Algebraic Techniques of Integration

Up to this point you have been finding antiderivatives of functions that followed directly from their derivative rules. Today we learn how to undo a function that used the chain rule to find the derivative. First, a quick review of the Chain Rule …

$$\frac{d}{dx}\left(f\left(g\left(x\right)\right)\right)=f'\left(g\left(x\right)\right)\cdot g'\left(x\right).$$

Outside derivative Inside derivative

The Chain Rule gives us a product of two factors: the "outside derivative" and the "inside derivative." So, if a function has this form then it has an antiderivative.

Example 1: Use the Chain Rule to differentiate $f(x)=\left(3x^4-5\right)^{12}$.

Example 2: Tell whether or not each antiderivative is going to undo a chain rule.

a) $\int\left(x^2-1\right)^3 2x\,dx$ b) $\int 3x^2\sqrt{x^3+2}\,dx$ c) $\int x\left(x^3-5\right)dx$

Substitution Method

Let u be the "inside function" and $du=u'\left(x\right)dx$

It is important to note, that with substitution, the goal is to substitute ALL values of the integrand with either *u* or *du*. Any "extras" must be accounted for, or substitution will NOT work!

<u>*Indefinite Integration with Substitution*</u>

Example 3: $\int x^3\sqrt{x^4+2}\,dx$

Example 4: $\int \sin^2\left(3x\right)\cos\left(3x\right)dx$

Example 5: $\int \frac{x}{x^2+2}\,dx$

Definite Integrals with Substitution

Indefinite Integrals needed a " +C " at the end of every antiderivative … Definite Integrals have limits.

If you change the variables, the limits still refer to the original variable … How will you decide to deal with those limits? … you have two choices …

#1: Leave the limits in terms of the original variable and integrate like you did for the indefinite integrals. Once you have returned all variables back to the original letter, you can plug in the upper limits and lower limits.

#2: Using the rule for the change of variables, change the limits with the same rule … then you never need to return to the original variable.

♬: THE LIMITS MUST MATCH THE VARIABLE BEING USED, OR THERE MUST BE SOME NOTATION TO INDICATE THAT THE LIMITS BEING USED ARE DIFFERENT FROM THE VARIABLE BEING USED!

Method 1: Use substitution to evaluate the integral, but do NOT change the upper and lower limits.

Example 6: Compute $\int_0^1 x\sqrt{1-x^2}\,dx$.

Method 2: Use substitution to evaluate the integral, and change the limits using the substitution rule you created.

Example 7: Evaluate $\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\,dx$.

Example 8: Compute $\int_0^{\pi/4} \frac{\tan^3\theta}{\cos^2\theta}\,d\theta$.

Algebraic Techniques

When substitution doesn't work, you sometimes just need to "massage" the problem into a form that will work using some algebraic techniques.

Long Division … when the numerator has a degree greater than or equal to the denominator

Example 9: $\int \frac{x^2-1}{x^2+1}\,dx$

Expand … when the "inside" doesn't have a derivative on the outside , try expanding the function

Example 10: $\int (\sin x + \cos x)^2\,dx$

Complete the Square … useful when you have a x^2 and x term in the denominator but no x term in the numerator.

Example 11: $\int \frac{2\,dx}{x^2-6x+10}$

Separate the numerator … when you have more than one term in the numerator

Example 12: $\int \frac{3x+2}{\sqrt{1-x^2}}\,dx$

Integrals of Trigonometric and Squared Trigonometric Functions

First, a few identities from trigonometry that you may or may not remember.

1. $1+\tan^2 x=\sec^2 x$ … everyone remembers this one, right?!

2. $\cos(2x)=2\cos^2 x-1$ … which can be rewritten to $\cos^2 x=\dfrac{1+\cos(2x)}{2}$

3. $\cos(2x)=1-2\sin^2 x$ … which can be rewritten to $\sin^2 x=\dfrac{1-\cos(2x)}{2}$

Can you integrate all of these functions? The first 4 should already be known.

$\int \sin x\, dx$ $\int \cos x\, dx$ $\int \sec^2 x\, dx$ $\int \csc^2 x\, dx$

$\int \tan x\, dx$ $\int \cot x\, dx$

$\int \csc x\, dx$ $\int \sec x\, dx$

$\int \sin^2 x\, dx$ $\int \cos^2 x\, dx$

$\int \tan^2 x\, dx$ $\int \cot^2 x\, dx$

Examples with Inverse Trigonometric Functions

Example 13: $\int \frac{dx}{\sqrt{4-x^2}}$

Example 14: $\int \frac{dx}{x\sqrt{4x^2-9}}$

6.4 EXPONENTIAL GROWTH AND DECAY

Notecards from Section 6.4: Derivation of an exponential function

In many applications, the rate of change of a variable *y* is proportional to the value of *y*. If *y* is a function of time *t*, we can express this statement as

Example 1: Find the solution to this differential equation given the initial condition that $y = y_0$ when $t = 0$.
(This is the derivation of an exponential function … see notecards … AND you want to know how to do it yourself!)

Exponential Growth and Decay Model

If *y* changes at a rate proportional to the amount present ($\frac{dy}{dt} = ky$) and $y = y_0$ when $t = 0$, then

$$y = y_0 e^{kt}$$

where *k* is the **proportional constant**.

Exponential **growth** occurs when $k > 0$, and exponential **decay** occurs when $k < 0$.

Example 2: The rate of change of *y* is proportional to *y*. When $t = 0$, $y = 2$. When $t = 2$, $y = 4$. What is the value of *y* when $t = 3$?

Example 3: **Newton's Law of Cooling**: Newton's Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature in the surrounding medium. A detective finds a murder victim at 9 am. The temperature of the body is measured at 90.3 °F. One hour later, the temperature of the body is 89.0 °F. The temperature of the room has been maintained at a constant 68 °F.

(a) Assuming the temperature, T, of the body obeys Newton's Law of Cooling, write a differential equation for T.

(b) Solve the differential equation to estimate the time the murder occurred.