5.4 FUNDAMENTAL THEOREM OF CALCULUS

Notecards from Section 5.4: Fundamental Theorem of Calculus Part 1 (Extended); Fundamental Theorem of Calculus Part 2 (Evaluation)

Do you remember the Fundamental Theorem of Algebra? Just thought I'd ask …

The Fundamental Theorem of Calculus has two parts. These two parts tie together the concept of integration and differentiation and is regarded by some to by the most important computational discovery in the history of mathematics! Be honest! … Deep down, you're not only really impressed, but you're filled with anticipation! I know it's hard, but try to contain your excitement!

Example 1: In section 5.2 we found an estimate for the distance traveled by finding the area between a velocity function and the *x*-axis. If *v* (*t*) is that velocity function (that is above the *x*-axis) and the time is from 0 to 10 seconds, how might we use calculus to define the distance traveled?

Example 2: Suppose a car's position is given by $s(t)=\frac{3}{2}t^2+30t+25$ where *t* is time in seconds, and $0 \le t \le 10$.

a) What is the position of the car at *t* = 0 seconds?

b) What is the position of the car at *t* = 10 seconds?

c) What is the change in position of the car from time *t* = 0 to time *t* = 10 seconds?

d) How does this question relate to the previous example?

The Fundamental Theorem of Calculus [The Evaluation Part]

If *f* is continuous at every point of [*a*, *b*],

$$\int_a^b f(x)\,dx = F(b)-F(a),$$

<u>where F(x) is an antiderivative of f (x).</u>

(… that last phrase is actually the toughest part! … below is a list of antiderivatives you should become familiar with)

The rules below are not formally stated until chapter 6, but knowing what you know about derivatives, you should be able to make the following connections:

Integral Formulas

1. Power Rule for x^n when $n \neq -1$: $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$
2. Rule for x^n when $n = -1$: $\int \frac{1}{x} \, dx = \ln|x| + C$
3. $\int e^{kx} \, dx = \frac{e^{kx}}{k} + C$
4. $\int \sin(kx) \, dx = \frac{-\cos(kx)}{k} + C$
5. $\int \cos(kx) \, dx = \frac{\sin(kx)}{k} + C$
6. $\int \sec^2(x) \, dx = \tan(x) + C$
7. $\int \csc^2(x) \, dx = -\cot(x) + C$
8. $\int \sec(x)\tan(x) \, dx = \sec(x) + C$
9. $\int \csc(x)\cot(x) \, dx = -\csc(x) + C$

Example 3: $\int_0^3 x^2 \, dx$

Example 4: $\int_{\pi/2}^{\pi} (1 + \cos x) dx$

Example 5: $\int_{-1}^{2} 3^x \, dx$

Example 6: $\int_4^9 f'(x) dx =$

This last example will be an extremely important concept as we go through the rest of the semester!

Using the evaluation part, we are going to develop the concept of the other part of the Fundamental Theorem of Calculus. You're book calls this *Part 1*, because it proves them in the opposite order. Our goal here isn't really to prove the Fundamental Theorem of Calculus, Part 1, but to understand how it works.

First, a quick overview …

#1. We are going to create a function that is defined as an integral … then,

#2: Using this function we are going to find the derivative of this function … thus tying the two concepts of calculus together forever!!!

Keep in mind that if we can define a function as an integral and take a derivative, then we can answer all the same types of questions about increasing, decreasing, concave up, concave down, and inflection points that we did earlier in the year. …you haven't forgotten all those reasons have you?

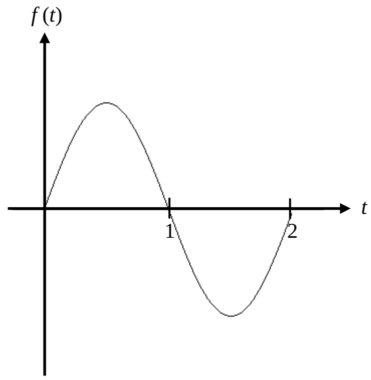
Step #1: So, to see how it is possible to define a function using an integral, consider the examples below.

The graph of $f(t)$ given below has odd symmetry and is periodic (with period = 2). Also, $\int_0^1 f(t)\,dt = \frac{4}{3}$.

Example 7: Let $F(x) = \int_0^x f(t)\,dt$.

a) Complete the following table:

x	$F(x)$
–1	
0	
1	
2	
3	

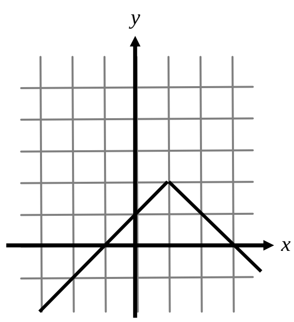


Example 8: Let $g(x) = \int_{-2}^x w(t)\,dt$, where the graph of $w(t)$ is given below.

a) Find $g(0)$.

b) Find $g(2)$.

c) Find $g(-3)$.



OK … now that we've established that we are able to define a function as an integral, let's talk about how to find a derivative of such a function. While interpreting a function defined as an integral is a valid skill in its own right, our goal here is to simply discover patterns found when taking the derivative.

Example 10: Suppose $g(x)=\int_0^x \cos t\,dt$. What is $g'(x)$?

a) Evaluate the right side of the equation.

b) Take the derivative.

Example 11: Suppose $g(x)=\int_{-2}^x \cos t\,dt$. What is $g'(x)$?

a) First … how is this problem different than the first one?

b) Evaluate the right side of the equation.

c) Take the derivative.

d) Did changing the lower limit from 0 to –2 matter? Explain.

Example 12: Suppose $g(x)=\int_a^x \cos t\,dt$, where *a* represents any constant. What is $g'(x)$?

Example 13: Suppose $g(x)=\int_x^a \cos t\,dt$, where *a* represents any constant. What is $g'(x)$?

Example 14: Let $g(x)=\int_1^x \sqrt{1+t^3}\, dt$. What is $g'(x)$?

a) Why is this example different from the previous examples?

b) Suppose the antiderivative of $\sqrt{1+t^3}$ is $h(t)$. This means $h'(t)=$ ______________.

c) Using *h* (*t*), evaluate the right side of the equation.

d) Take the derivative.

Since example 10, the integral portion had an upper limit of *x* and a lower limit of a constant. In this scenario, we have …

The Fundamental Theorem of Calculus [Part #1 … Simple]

$$\frac{d}{dx}\left[\int_a^x f(t)\, dt\right]=f(x)$$

In your own words, describe what happens when we take a derivative of a definite integral from a constant to *x*.

Example 15: If $g(x)=\int_{-2}^x w(t)dt$, then $g'(x)=$?

Example 16: $\frac{d}{dx}\left[\int_3^x \left(5t^2-6t+1\right) dt\right]$

How is the next example different?

Example 17: $\dfrac{d}{dx}\left[\displaystyle\int_{2}^{3x^2}\left(5t^2-6t+1\right)dt\right]$

Example 18: Suppose $g(x)=\displaystyle\int_{5x}^{3x^2}\sqrt{1+t^3}\,dt$. Find $g'(x)$.

Notice any patterns? …
In these last two examples, the upper and lower limits have been functions of *x*, which means the derivatives have used the chain rules. In this scenario, we have …

The Fundamental Theorem of Calculus [Part 1 … Extended]

$$\frac{d}{dx}\left[\int_{v(x)}^{u(x)} f(t)\,dt\right]=f\left(u(x)\right)\cdot u'(x)-f\left(v(x)\right)\cdot v'(x)$$

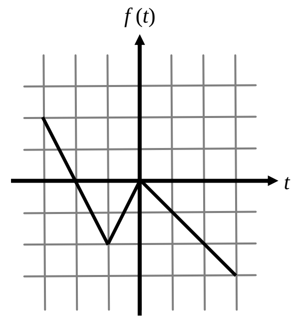
Example 19: Find $\dfrac{d}{dx}\left[\displaystyle\int_{x^2}^{3x} f(t)\,dt\right]$

Putting it all together …

Example 20: Suppose the function below is the graph of $f(t)$ and $g(x) = \int_{-1}^{x} f(t)\,dt$.

a) Complete the table:

x	***g* (*x*)**
–3	
–2	
–1	
0	
1	
2	
3	



b) What are the intervals on which *g* is increasing or decreasing? Justify each response.

c) What are the intervals on which *g* is concave up or concave down? Justify each response.

d) For what value of *x* does *g* have a relative maximum? Justify your response.

e) For what value of *x* does *g* have an inflection point? Justify your response.

f) Graph *g* (*x*)