5.4 FUNDAMENTAL THEOREM OF CALCULUS

Do you remember the Fundamental Theorem of Algebra? Just thought I'd ask …

The Fundamental Theorem of Calculus has two parts. These two parts tie together the concept of integration and differentiation and is regarded by some to by the most important computational discovery in the history of mathematics! Be honest! … Deep down, you're not only really impressed, but you're filled with anticipation! I know it's hard, but try to contain your excitement!

In §5.1, we saw how the area under a velocity curve can be used to determine distance traveled. Using this same idea, complete the following examples:

Example: A car is traveling so that its speed is never decreasing during a 10 – second interval. The speed at various moments in time is listed in the table below.

Time (sec)	0	2	4	6	8	10
Speed (ft/sec)	30	36	40	48	54	60

a) Use the table to help explain why the best lower estimate for the distance traveled in the first 2 seconds is 60 feet.

b) Use the table to help explain why the best upper estimate for the distance traveled in the first 2 seconds is 72 feet.

c) Use the table to give the best lower estimate for the distance traveled in the first 10 seconds.
♬ : An answer of 300 feet (which ignores some of the data) is not correct.

d) Use the table to give the best upper estimate for the distance traveled in the first 10 seconds.

… These sums of products that you have found in *c* and *d* are called __________________.

e) If you choose the lower estimate for your approximation of how far the car travels, what is the maximum amount your approximation could differ from the exact distance?

f) Choose speeds to correspond with t = 1, 3, 5, 7, and 9 seconds. Keep the nondecreasing nature of the above table and do not select the average of the consecutive speeds. Find new best upper and lower estimates for the distance traveled for these 10 seconds.

Time (sec)	0	1	2	3	4	5	6	7	8	9	10
Speed (ft/sec)	30		36		40		48		54		60

g) Write an expression giving the ACTUAL distance this car traveled in 10 seconds, if it's velocity was $v(t)$.

If you felt like we haven't done anything new yet, then good! … If you felt like this was the first time you've done anything like what we just did, then I hope you understand it.

Example: Suppose you have a car whose position is given by $s(t) = t^2 + t + 20$ where t is time in seconds, and $0 \le t \le 10$.

a) What is the position of the car at $t = 0$ seconds?

b) What is the position of the car at $t = 10$ seconds?

c) What is the change in position of the car from time $t = 0$ to time $t = 10$ seconds?

d) How does this question relate to the previous page?

The Fundamental Theorem of Calculus [The Evaluation Part]

If f is continuous at every point of $[a, b]$,

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$.

(… that last phrase is actually the toughest part!)

Example: $\int_0^3 x^2 \, dx$

Example: $\int_{\pi/2}^{\pi} (1 + \cos x)\,dx$

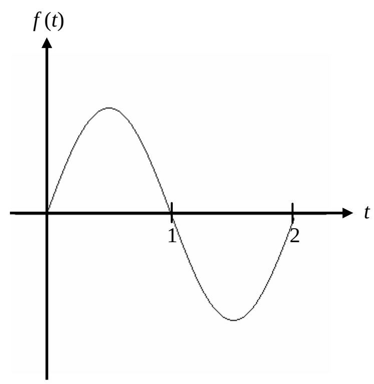
Example: $\int_{-1}^{2} 3^x \, dx$

Using the evaluation part, we are going to develop the concept of the other part of the Fundamental Theorem of Calculus. You're book calls this *Part 1*, because it proves them in the opposite order. Our goal here isn't really to prove the Fundamental Theorem of Calculus, Part 1, but to understand how it works.

First, a quick overview … we are going to create a function that is defined as an integral … then, using this function we are going to find the derivative of this function … thus tying the two concepts of calculus together forever!!! Keep in mind that if we can define a function as an integral and take a derivative, then we can answer all the same types of questions about increasing, decreasing, concave up, concave down, and inflection points that we did earlier in the year.

So, to see how it is possible to define a function using an integral, consider the examples below.

Example: The graph of $f(t)$ given below has odd symmetry around the point (2, 0). On the interval [0, 2], the graph is symmetric with respect to the line $t = 1$. Also, $\int_0^1 f(t)\,dt = \frac{4}{3}$.



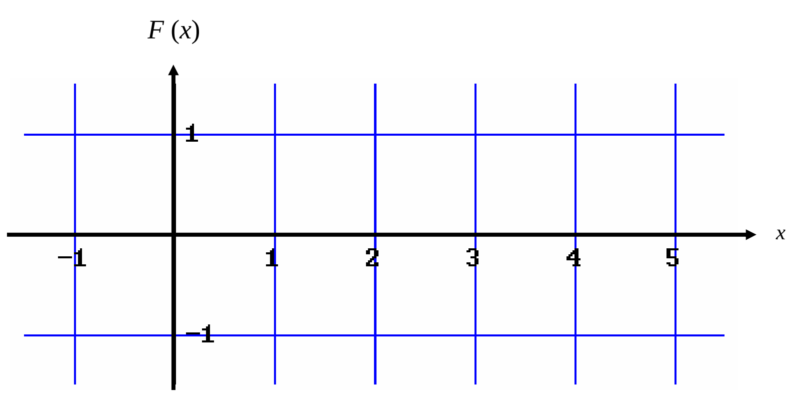
OK … now to define the function as an integral.

Example: Let $F(x) = \int_0^x f(t)\,dt$.

a) Complete the following table:

x	0	1	2	3	4
F (x)					

b) Sketch your best estimate of the graph of F on the grid below.



Example: Suppose $g(x)=\int_{0}^{x}\sin t\,dt$. What is $g'(x)$?

a) Evaluate the right side of the equation.

b) Take the derivative.

Example: Suppose $g(x)=\int_{-2}^{x}\sin t\,dt$. What is $g'(x)$?

a) First … how is this problem different than the first one?

b) Evaluate the right side of the equation.

c) Take the derivative.

d) Did changing the lower limit from 0 to –2 matter? Explain.

Example: Suppose $g(x)=\int_{a}^{x}\sin t\,dt$, where a represents any constant. What is $g'(x)$?

Example: Suppose $g(x)=\int_{x}^{a}\sin t\,dt$, where a represents any constant. What is $g'(x)$?

Example: Let $g(x)=\int_{1}^{x}\sqrt{1+t^3}\,dt$. What is $g'(x)$?

a) Why is this example different from the previous examples?

b) Suppose the antiderivative of $\sqrt{1+t^3}$ is $h(t)$. This means $h'(t)=$ ______________.

c) Using $h(t)$, evaluate the right side of the equation.

d) Take the derivative.

Example: Try again with $g(x)=\int_{0}^{3x^2}\sqrt{1+t^3}\,dt$. Find $g'(x)$.

Example: Find $\frac{d}{dx}\left[\int_{1}^{\sin x}\sqrt{1+t^3}\,dt\right]$.

Example: Find $\frac{d}{dx}\left[\int_{x^2}^{3x} f(t)\,dt\right]$

Example: Find $\frac{d}{dx}\left[\int_{\sin x}^{x^3} f(t)\,dt\right]$

Example: Find $\frac{d}{dx}\left[\int_{\sin x}^{x^3} e^{t^2}\,dt\right]$

Notice any patterns? …

The Fundamental Theorem of Calculus [Part #1 … Simple]

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

In other words, the Integral and the Derivative are just __________________________.

Example: $\frac{d}{dx}\left[\int_3^x \left(5t^2 - 6t + 1\right) dt\right]$

The Fundamental Theorem of Calculus [Part 1 … Extended]

$$\frac{d}{dx}\left[\int_{v(x)}^{u(x)} f(t)\,dt\right] = f\left(u(x)\right)\cdot u'(x) - f\left(v(x)\right)\cdot v'(x)$$

Example: $\frac{d}{dx}\left[\int_{x^3}^{4x} \left(5t^2 - 6t + 1\right) dt\right]$

Notecards from Section 5.4: Fundamental Theorem of Calculus Part 1 (Extended); Fundamental Theorem of Calculus Part 2 (Evaluation)