5.2 DEFINITE INTEGRALS

Notecards from Section 5.2: Riemann Sum; Definition of Definite Integral; Properties of a Definite Integral (more in the next section)

Riemann Sums

In the last section we found the area under a curve by finding the area of a finite number of rectangles (LRAM, RRAM, and MRAM) or a finite number of trapezoids (TRAPEZOID RULE). Every one of these was an example of what is called a **Riemann Sum**. We're going to stick with RECTANGLES for the time being.

The following steps illustrate what has to happen in order for the sum to be considered a RIEMANN Sum.

Step 1: Start with a *continuous* function on a *closed* interval.

Step 2: **_Partition_** the interval into *n* subintervals. The kth subinterval has width $\triangle x_k$.
(We made sure they were all the same size, but it turns out, it really doesn't matter for what we're doing here.)

Step 3: In each subinterval, pick **_any_** number and call the number picked from the *k*th subinterval c_k
(LRAM picked the left endpoint … RRAM picked the right endpoint … MRAM picked the midpoint)

Step 4: For each interval, using the width, $\triangle x_k$, of the interval as the base, create a rectangle that extends from the x – axis to the function value, $f(c_k)$, of the number you picked in each interval.
(♪ : Some of these rectangles could lie below the x – axis.)

Step 5: On each interval, form the product $f(c_k)\cdot \triangle x_k$.
(If all our rectangles lie above the *x*-axis, this would have been the area of each rectangle)

Step 6: Find the SUM of each of these products.

$$f(c_1)\cdot \triangle x_1 + f(c_2)\cdot \triangle x_2 + f(c_3)\cdot \triangle x_3 + f(c_4)\cdot \triangle x_4 + \cdots + f(c_n)\cdot \triangle x_n = \sum_{k=1}^{n} f(c_k)\cdot \triangle x_k \ .$$

Following these steps gives you a **Riemann Sum for *f* on the interval [*a*, *b*]**. Every Riemann sum depends on the partition you choose (i.e. the number of subintervals) and your choice of the number within each interval, c_k.

Definite Integral as a Limit of a Riemann Sum

Our Goal … Develop a Mathematical DEFINITION of a Definite Integral … (i.e. What is a Definite Integral?)

While "AREA" is inherently positive, a Riemann sum can be negative if the rectangles lie below the *x*-axis.

A DEFINITE INTEGRAL is defined as a Limit of a Riemann Sum.

Option #1: If you noticed in step 2 above, we didn't care if our subintervals were the same width. If we use the notation $\|P\|$ to denote the longest subinterval length we can force the longest subinterval length to 0 using a limit of the Riemann Sum as follows:

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k$$

Option #2: If we make sure the subintervals are all the same width, we can increase the number of rectangles to infinity using a limit of the Riemann Sum as follows:

$$\lim_{n\to \infty} \sum_{k=1}^{n} f(c_k)\cdot \Delta x$$

Notation for Definite Integrals

The limit notation we used last is the form we will use to develop Integral notation. As the number of rectangles goes to infinity, the width of each rectangle, Δx, goes to zero. As we did in the section on differentials, we are going to use the notation dx to represent this infinitely tiny distance.

The summation notation of sigma is going to be replaced with an *Integral Sign* , $\int$, which looks somewhat like a giant "S" for sum.

The $f\left(c_k\right)$ which represented a different function value for each interval is going to be replaced with $f(x)$ since the $x-$ values are going to be soooooooo close together it's almost as if we are evaluating the function at EVERY $x-$ value in the interval $[a, b]$. Combining all of this we have the following notation:

$$\int_a^b f(x)\, dx$$

We read the notation above as "***The Integral of f of x from a to b***"

Important ♪ (Actually it's a Theorem): IF the function is continuous , THEN the Definite Integral will exist. However, the converse, while true some of the time is NOT ALWAYS true.

Using Definite Integrals as Area

We can define the **area under the curve $y = f(x)$ from a to b** as an *integral* from a to b …
  … AS LONG AS THE CURVE IS NONNEGATIVE AND INTEGRABLE on the closed interval $[a, b]$.

Drawing a picture and using geometry is still a valid method of finding areas in this class!

Example 1: For each of the following examples, sketch a graph of the function, shade the area you are trying to find, then use geometric formulas to evaluate each integral.

a) $\displaystyle\int_2^9 3\, dx$

b) $\displaystyle\int_{-2}^1 |x|\, dx$.

c) $\displaystyle\int_{-3}^3 \sqrt{9 - x^2}\, dx$

So … what happens if the "area" is below the x-axis … as I mentioned before, "area" is inherently positive, but a Riemann sum … and therefore an Integral can have negative values if the curve lies below the x-axis.

Example 2: Consider the function $f(x) = 3 - x$. Sketch a graph of this function.

a) What is the "AREA" between the curve and the x – axis between $x = 4$ and $x = 8$?

b) Evaluate $\displaystyle\int_4^8 (3 - x)\, dx$

Example 3: Given $\displaystyle\int_0^\pi \sin x\, dx = 2$, use what you know about a sine function to evaluate the following integrals.

a) $\displaystyle\int_\pi^{2\pi} \sin x\, dx$

b) $\displaystyle\int_0^{2\pi} \sin x\, dx$

c) $\displaystyle\int_0^{\pi/2} \sin x\, dx$

d) $\displaystyle\int_{-\pi}^\pi \sin x\, dx$

e) $\displaystyle\int_0^\pi (2 + \sin x)\, dx$

The FnInt Function of your TI – 83+

By this point, hopefully you understand the following concepts:

      1. The limit of a Riemann Sum is used to define a Definite Integral
      2. A Definite Integral can be used to find the Area under a curve if the curve is above the x – axis, and if the curve is below the x – axis the value of the definite integral is "negative area" … even though no one in their right mind would ever actually use that phrase in a math class if they wanted to be taken seriously!
      3. Since the Definite Integral can be thought of as Area, you can draw a picture and use geometric formulas to find the areas.

BUT … what happens if you were asked right now, this instant, today to find the definite integral of a function that doesn't lend itself to nice geometric shapes?

The good news for now, is you don't even have to worry about how to do these by hand … YET!
You get to use your calculator! … At least until the next lesson ☺

How to Use Your Calculator to Find a Definite Integral:

The syntax for using your calculator is as follows: fnInt(*function*, *x*, *lower bound*, *upper bound*)
      1. Press MATH
      2. Press 9: fnInt(
      3. Follow the syntax above
            … enter the function, x, lower bound, upper bound (be sure to enter a comma between each)

Example 4: Evaluate $\displaystyle\int_{1}^{3} \left(x^2 - 2x + 2\right) dx$

Example 5: Evaluate $\displaystyle 3 + 2\int_{0}^{\pi/3} \tan x\, dx$

You can also do the same thing from the graphing screen.

Example 6: Graph $y = \sqrt{x}$ on a standard viewing window. Evaluate $\displaystyle\int_{1}^{8} \sqrt{x}\, dx$.

      Press 2nd TRACE (which is CALC), 7: ∫ f(x)dx, enter Lower Bound as 1, enter Upper Bound as 8.

      … the down side to using this method is that you MUST be able to set your window to SEE everything.