5.1 Estimating with Finite Sums

5.1 ESTIMATING WITH FINITE SUMS

| Notecards from Section 5.1 & 5.5: LRAM, RRAM, MRAM, Trapezoidal Rule |

Example 1: Suppose from the 2nd to 4th hour of your road trip, you travel with the cruise control set to exactly 70 miles per hour for that two hour stretch. How far have you traveled during this time?

Example 2: Sketch a graph modeling the situation in the above example. Geometrically, how can we indicate the total distance traveled?

Example 3: What if the velocity was NOT constant. Say, for instance the velocity in miles per hour is given by the function $v = 10t - t^2$, where $t$ is in hours, and we wanted to know the total distance traveled during the first 10 hours. Sketch this graph below. Geometrically speaking, do you think we can find the total distance traveled in the same was as before? Why or why not?

Word of caution to those brave few who are actually reading this … The following paragraphs are extremely important to the conceptual understanding of what we are about to do in Calculus. However, since you really haven't done anything yet, it might make you a little dizzy at first, so come back and read it again later. If you're reading this for the first time, you might experience that same feeling you get when you've been on the Tilt – A – Whirl one too many times at the fair. (Never been on the Tilt – A – Whirl? … well, take my word for it, it's not something you want to ride 10 minutes after eating a corn dog and a funnel cake!) Well, I warned you, but you've kept on reading anyway, so here it goes …

The key to finding the total distance traveled in the last example in a method similar to the first example is to break the time intervals into such short segments, that the velocity over those time segments is almost constant (this will require quite a few intervals). If the velocity is almost constant for each time interval, then we can find the distance traveled for each time interval (which is just the area of an extremely thin rectangle) and add all the areas of all the rectangles together. Sounds simple enough, right? Can you guess what extremely important calculus concept is involved?

We will spend MUCH more time with this later, but it turns out that if we are given the graph of a rate of change (like velocity in miles per hour) we will be able to find the total accumulated change over an interval (like total distance traveled, in miles) by finding the area under the curve.

OK, that last paragraph or two may not have made perfect sense to you … YET! … For now, THE GOAL is to figure out a way to find the area under the curve. This chapter actually discusses 5 ways to approximate this area, LRAM, RRAM, MRAM, Trapezoid Rule, and Simpson’s Rule, but we're only going to deal with 4 of them. (Simpson’s Rule is not in the AP curriculum)
5.1 Estimating with Finite Sums

The Area Problem and the Rectangular Approximation Method (RAM) (a.k.a. Riemann Sums)

Suppose we wanted to know the area of the region bounded by a curve, the $x$ – axis, and the lines $x = a$ and $x = b$, as shown at the right.

The first step is to divide the interval from $a$ to $b$ into subintervals. (The examples below show 4 and 8 subintervals, respectively.)

After dividing the given interval into subintervals, we can then draw rectangles using the width of each subinterval as the base.

The height of each rectangle is determined by the function value at a point in the specific subinterval, and can be determined using 3 different methods.

We could use the left endpoint of each subinterval (called LRAM), the right endpoint of each subinterval (RRAM), or the midpoint of each subinterval (MRAM).

Example 4: Which method is shown in the two graphs below?

Example 5: The total area under the curve then is approximately equal to the total area of all the rectangles. Which of the graphs above gives a better approximation of the area under the curve? Why? How could it be further improved?

Summary of the Process: A sketch is almost mandatory!

Step 1: Divide (or Partition) the interval into $n$ subintervals.

Step 2: Create $n$ rectangles whose base equals the width of each subinterval and whose height is determined by the function value at the left endpoint, the right endpoint, or the midpoint of the subinterval.

Step 3: Find the area of all $n$ rectangles and add them together.

Example 6: Illustrate the use of RRAM and MRAM on the graphs below. (use 4 rectangles)
Example 7: Use 4 rectangles to approximate the area under the graph of \( y = x^2 - 2x + 2 \) from \( x = 1 \) to \( x = 3 \). Use LRAM, RRAM, and then MRAM.

\[
\begin{array}{cccc}
  \hline
  \text{Time} & \text{Velocity} & \text{Time} & \text{Velocity} \\
  \text{(sec)} & \text{(in./sec)} & \text{(sec)} & \text{(in./sec)} \\
  \hline
  0 & 0 & 6 & 11 \\
  1 & 12 & 7 & 6 \\
  2 & 22 & 8 & 2 \\
  3 & 10 & 9 & 6 \\
  4 & 5 & 10 & 0 \\
  5 & 13 & & \\
  \hline
\end{array}
\]

Example 8: Using your rectangles as a guide, find each approximation.

a) LRAM = 

b) RRAM = 

c) MRAM = 

Example 9: It is not necessary to have a graph to estimate the area. Suppose the table below shows the velocity of a model train engine moving along a track for 10 seconds.

a) Using a left Riemann Sum with 10 subintervals, estimate the distance traveled by the engine in the first 10 seconds.

b) Using a Midpoint Riemann Sum with 5 subintervals, estimate the distance traveled by the engine in the first 10 seconds.
5.1 Estimating with Finite Sums

The Trapezoidal Rule (Really §5.5)

While rectangles make a fairly good approximation, it's easy to see that we're going to need a lot of them to provide a good estimate. We can find a better estimate in less time if we use trapezoids. If we were to partition the interval into subintervals like we did before, we can use each subinterval to create a trapezoid if we just connect the function values of the left and right endpoints. Before we begin, let's make sure you understand the area formula for a trapezoid.

Area of a Trapezoid: \( A = \frac{1}{2} \cdot h \cdot (b_1 + b_2) \)

While not all trapezoids must look like this, the one's we're going to be using will, so we'll stick with this picture. Label all the parts of the area formula on the picture below.

The biggest difference will be the orientation of the trapezoid. The ones we are going to be drawing will look like

Draw a set of axes on the picture above and a function that goes through the top left and top right points of the trapezoid. The "height" of the trapezoid is just the width of a subinterval, and the "bases" are going to be the function values of the left and right endpoints.

Example 10: Use 4 trapezoids to approximate the area under the curve \( y = x^2 - 2x + 2 \) from \( x = 1 \) to \( x = 3 \). Sketch the trapezoids.

By the way … The trapezoid rule connects the left and right hand endpoints with a segment. This method of approximation turns out to be a pretty good, but if you were to connect the endpoints with a curve (namely a parabola) the approximation would be even better. Connecting the endpoints with a parabola and finding the area of the resulting shape is the basis behind the fifth method of approximation called Simpson's Rule. You can read about it in your book if you find yourself just dying of curiousity, but it's not in the AP curriculum.
5.2 Definite Integrals

5.2 DEFINITE INTEGRALS

Notecards from Section 5.2: Riemann Sum; Definition of Definite Integral; Properties of a Definite Integral (more in the next section)

Riemann Sums

In the last section we found the area under a curve by finding the area of a finite number of rectangles (LRAM, RRAM, and MRAM) or a finite number of trapezoids (TRAPEZOID RULE). Every one of these was an example of what is called a Riemann Sum. We’re going to stick with RECTANGLES for the time being.

The following steps illustrate what has to happen in order for the sum to be considered a Riemann Sum.

Step 1: Start with a continuous function on a closed interval.

Step 2: Partition the interval into \( n \) subintervals. The \( k^{th} \) subinterval has width \( \Delta x_k \).
(We made sure they were all the same size, but it turns out, it really doesn’t matter for what we’re doing here.)

Step 3: In each subinterval, pick any number and call the number picked from the \( k^{th} \) subinterval \( c_k \).
(LRAM picked the left endpoint … RRAM picked the right endpoint … MRAM picked the midpoint)

Step 4: For each interval, using the width, \( \Delta x_k \), of the interval as the base, create a rectangle that extends from the \( x \)-axis to the function value, \( f(c_k) \), of the number you picked in each interval.
(\( \therefore \) Some of these rectangles could lie below the \( x \)-axis.)

Step 5: On each interval, form the product \( f(c_k) \cdot \Delta x_k \).
(If all our rectangles lie above the \( x \)-axis, this would have been the area of each rectangle)

Step 6: Find the SUM of each of these products.

\[
\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k
\]

Following these steps gives you a Riemann Sum for \( f \) on the interval \([a, b]\). Every Riemann sum depends on the partition you choose (i.e. the number of subintervals) and your choice of the number within each interval, \( c_k \).

Definite Integral as a Limit of a Riemann Sum

Our Goal … Develop a Mathematical DEFINITION of a Definite Integral … (i.e. What is a Definite Integral?)

While “AREA” is inherently positive, a Riemann sum can be negative if the rectangles lie below the \( x \)-axis.

A DEFINITE INTEGRAL is defined as a Limit of a Riemann Sum.

Option #1: If you noticed in step 2 above, we didn’t care if our subintervals were the same width. If we use the notation \( \|P\| \) to denote the longest subinterval length we can force the longest subinterval length to 0 using a limit of the Riemann Sum as follows:

\[
\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k
\]

Option #2: If we make sure the subintervals are all the same width, we can increase the number of rectangles to infinity using a limit of the Riemann Sum as follows:

\[
\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x
\]
5.2 Definite Integrals

**Notation for Definite Integrals**

The limit notation we used last is the form we will use to develop integral notation. As the number of rectangles goes to infinity, the width of each rectangle, \( \Delta x \), goes to zero. As we did in the section on differentials, we are going to use the notation \( dx \) to represent this infinitely tiny distance.

The summation notation of sigma is going to be replaced with an Integral Sign, \( \int \), which looks somewhat like a giant "S" for sum.

The \( f_k \) which represented a different function value for each interval is going to be replaced with \( f(x) \) since the \( x \)–values are going to be sooooooo close together it's almost as if we are evaluating the function at EVERY \( x \)–value in the interval \([a, b]\). Combining all of this we have the following notation:

\[
\int_a^b f(x) \, dx
\]

We read the notation above as "The Integral of \( f \) of \( x \) from \( a \) to \( b \)"

**Important \( \sharp \) (Actually it's a Theorem):** IF the function is continuous, THEN the Definite Integral will exist. However, the converse, while true some of the time is NOT ALWAYS true.

**Using Definite Integrals as Area**

We can define the area under the curve \( y = f(x) \) from \( a \) to \( b \) as an integral from \( a \) to \( b \) … … AS LONG AS THE CURVE IS NONNEGATIVE AND INTEGRABLE on the closed interval \([a, b]\).

Drawing a picture and using geometry is still a valid method of finding areas in this class!

**Example 1:** For each of the following examples, sketch a graph of the function, shade the area you are trying to find, then use geometric formulas to evaluate each integral.

a) \( \int_2^9 3 \, dx \)

b) \( \int_{-2}^1 |x| \, dx \)

c) \( \int_{-3}^3 \sqrt{9-x^2} \, dx \)
So … what happens if the “area” is below the $x$-axis … as I mentioned before, “area” is inherently positive, but a Riemann sum … and therefore an Integral can have negative values if the curve lies below the $x$-axis.

**Example 2:** Consider the function $f(x) = 3 - x$. Sketch a graph of this function.

<table>
<thead>
<tr>
<th>a)</th>
<th>What is the &quot;AREA&quot; between the curve and the $x$-axis between $x = 4$ and $x = 8$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>b)</td>
<td>Evaluate $\int_{4}^{8} 3 - x , dx$</td>
</tr>
</tbody>
</table>

**Example 3:** Given $\int_{0}^{\pi} \sin x \, dx = 2$, use what you know about a sine function to evaluate the following integrals.

<table>
<thead>
<tr>
<th>a) $\int_{0}^{2\pi} \sin x , dx$</th>
<th>b) $\int_{0}^{2\pi} \sin x , dx$</th>
<th>c) $\int_{0}^{\pi} \sin x , dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>d) $\int_{-\pi}^{\pi} \sin x , dx$</td>
<td>e) $\int_{0}^{\pi} 2 + \sin x , dx$</td>
<td></td>
</tr>
</tbody>
</table>
5.2 Definite Integrals

The \textit{FnInt} Function of your TI – 83+

By this point, hopefully you understand the following concepts:

1. The limit of a Riemann Sum is used to define a Definite Integral
2. A Definite Integral can be used to find the Area under a curve if the curve is above the \(x\) – axis, and if the curve is below the \(x\) – axis the value of the definite integral is "negative area" … even though no one in their right mind would ever actually use that phrase in a math class if they wanted to be taken seriously!
3. Since the Definite Integral can be thought of as Area, you can draw a picture and use geometric formulas to find the areas.

BUT … what happens if you were asked right now, this instant, today to find the definite integral of a function that doesn't lend itself to nice geometric shapes?

The good news for now, is you don't even have to worry about how to do these by hand … YET!
You get to use your calculator! … At least until the next lesson 😊

How to Use Your Calculator to Find a Definite Integral:

The syntax for using your calculator is as follows: \( \text{fnInt}( \text{function}, x, \text{lower bound}, \text{upper bound}) \)

1. Press MATH
2. Press 9: fnInt(
3. Follow the syntax above … enter the function, \(x\), lower bound, upper bound (be sure to enter a comma between each)

\textit{Example 4:} Evaluate \( \int_{1}^{3} x^2 - 2x + 2 \, dx \)

\textit{Example 5:} Evaluate \( 3 + 2 \int_{0}^{\pi} \tan x \, dx \)

You can also do the same thing from the graphing screen.

\textit{Example 6:} Graph \( y = \sqrt{x} \) on a standard viewing window. Evaluate \( \int_{1}^{3} \sqrt{x} \, dx \).

Press 2nd TRACE (which is CALC), 7: \( \int f(x) \, dx \), enter Lower Bound as 1, enter Upper Bound as 8.

… the down side to using this method is that you MUST be able to set your window to SEE everything.
5.3 Definite Integrals and Antiderivatives

5.3 DEFINITE INTEGRALS AND ANTIDERIVATIVES

Notecards from Section 5.3: Properties of a Definite Integral; Average Value of f(x); Relationship between Average Value of f(x) and Average Rate of Change

In the last section we defined the definite integral as a limit of a Riemann Sum, thus we can use the properties of limits to develop properties of the definite integral. The proofs of each of the rules below are derived directly from the properties of limits and Riemann Sums.

Rules for Definite Integrals

1. Order of Integration: \[ \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \]
   If you reverse the order of integration you get the opposite answer.

2. Zero: \[ \int_{a}^{a} f(x) \, dx = 0 \]
   This should make sense if you think about the "area" of a rectangle with no width.

3. Constant Multiple: If \( k \) is any constant, then \[ \int_{a}^{b} k \cdot f(x) \, dx = k \cdot \int_{a}^{b} f(x) \, dx \]
   Taking the constant out of the integral many times makes it simpler to integrate.

4. Sum and Difference: \[ \int_{a}^{b} [f(x) \pm g(x)] \, dx = \int_{a}^{b} f(x) \, dx \pm \int_{a}^{b} g(x) \, dx \]
   This allows you to integrate functions that are added or subtracted separately. Notice, there are NO rules here for two functions that are multiplied or divided … that comes later!

5. Additivity: \[ \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx = \int_{a}^{c} f(x) \, dx \]
   Pay close attention to the limits of integration … this comes in handy when dealing with total area or other functions where we need to break them into smaller parts.

Example 1: Given \( \int_{2}^{6} f(x) \, dx = 10 \) and \( \int_{2}^{6} g(x) \, dx = -2 \), find the following:

a) \[ \int_{2}^{6} [f(x) + g(x)] \, dx \]

b) \[ \int_{2}^{6} [g(x) - f(x)] \, dx \]

c) \[ \int_{2}^{6} 3f(x) \, dx \]

d) \[ \int_{2}^{6} f(x) + 2 \, dx \]
Example 2: Given \( \int_0^5 f \ x \ dx = 10 \) and \( \int_5^7 f \ x \ dx = 3 \), find the following:

a) \( \int_0^7 f \ x \ dx \)

b) \( \int_0^5 f \ x \ dx \)

c) \( \int_5^5 f \ x \ dx \)

d) \( \int_0^5 3f \ x \ dx \)

Average Value of a Function

Suppose you wanted to find the average temperature during a 24 hour period. How could you do it?

Suppose \( f(t) \) represents the temperature at time \( t \), measured in hours since midnight. One way to start is to measure the temperature at \( n \) equally spaced times \( t_1, t_2, t_3, \ldots, t_n \) and then average those temperatures.

Example 3: Using this method, write an expression for the AVERAGE temperature.

The larger the number of measurements, the more accurately this will reflect the average temperature. Notice we can write this expression as a Riemann sum by first noting that the interval between measurements will be \( \Delta t = \frac{24}{n} \), so \( n = \frac{24}{\Delta t} \).

Example 4: Substitute this value of \( n \) into your expression above and simplify.

Example 5: The last expression gives us an approximate Average Temperature. As \( n \to \infty \) (meaning we are taking a lot of temperature readings) this Riemann Sum becomes a definite integral. Write the Definite Integral that gives us the Average Temperature since midnight.

Example 6: Do you think that there is any point during the day that the temperature reading on the thermometer is the exact value of the average temperature?
5.3 Definite Integrals and Antiderivatives

The process that we just used to find the average temperature is used to find the Average Value of any function.

**The Average Value of a Function**

If \( f \) is integrable on \([a, b]\), its **average value** on \([a, b]\) is given by

\[
\text{AVERAGE VALUE} = \frac{1}{b-a} \int_a^b f(x) \, dx = \frac{\int_a^b f(x) \, dx}{b-a}
\]

The *average value of a function* is just … “the integral over the interval”.

To get a more geometric idea of what the average value is, complete the following examples:

**Example 7:** Graph the function \( y = x^2 \) on \([0, 3]\) on the grid to the right.

**Example 8:** Set up a definite integral to find the average value of \( y \) on \([0, 3]\), then use your calculator to evaluate the definite integral.

**Example 9:** Graph this as a value as a function on the grid to the right. Does the function ever actually equal this value? If so, at what point(s) in the interval does the function assume its average value?

**Example 10:** What do you suppose is the relationship between the area between the \( x \)-axis and the curve \( y = x^2 \) on \([0, 3]\) and the area of the rectangle formed using the average value as the height and the interval \([0, 3]\) as the width?

**The Mean Value Theorem for Definite Integrals** … see the connection … Mean … Average …

The Mean Value Theorem for Integrals basically says that if you are finding the area under a curve between \( x = a \) and \( x = b \), then there is some number \( c \) between \( a \) and \( b \) whose function value you can use to form a rectangle that has an area equal to the area under the curve.

**Example 11:** What is an expression that could be used to determine the area under the curve from \( a \) to \( b \) ?

**Example 12:** What is the area of the shaded rectangle?

This value of \( f(c) \) is just the Average Value of \( f \) on the interval \([a, b]\).
So another way to look at this is the Mean Value Theorem for Integrals just says that at some point within the interval the function MUST equal its average value.

**Mean Value Theorem for Integrals**

If $f$ is continuous on $[a, b]$, then at some point $c$ in $[a, b]$,

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx$$


...Once again, we have a theorem that tells us a value of $c$ exists, but the theorem doesn’t actually find it for us!

It is greatly important that you understand the difference between *average rate of change* and *average value*.

More on this after we finish §5.4. For now, understand that the average rate of change is simply the “slope between two points” on a given function and the average value of the function is the “integral divided by the interval”.


5.4 Fundamental Theorem of Calculus

5.4 FUNDAMENTAL THEOREM OF CALCULUS

Do you remember the Fundamental Theorem of Algebra? Just thought I'd ask …

The Fundamental Theorem of Calculus has two parts. These two parts tie together the concept of integration and differentiation and is regarded by some to by the most important computational discovery in the history of mathematics! Be honest! … Deep down, you're not only really impressed, but you're filled with anticipation! I know it's hard, but try to contain your excitement!

Example 1: In section 5.2 we found an estimate for the distance traveled by finding the area between a velocity function and the x-axis. If \( v(t) \) is that velocity function (that is above the x-axis) and the time is from 0 to 10 seconds, how might we use calculus to define the distance traveled?

Example 2: Suppose a car's position is given by \( s(t) = \frac{1}{2}t^2 + 30t + 25 \) where \( t \) is time in seconds, and \( 0 \leq t \leq 10 \).

a) What is the position of the car at \( t = 0 \) seconds?

b) What is the position of the car at \( t = 10 \) seconds?

c) What is the change in position of the car from time \( t = 0 \) to time \( t = 10 \) seconds?

d) How does this question relate to the previous example?

The Fundamental Theorem of Calculus [The Evaluation Part]

If \( f \) is continuous at every point of \([a, b]\),

\[
\int_a^b f(x) \, dx = F(b) - F(a),
\]

where \( F(x) \) is an antiderivative of \( f(x) \).

( … that last phrase is actually the toughest part! … a list of antiderivatives follows that you should become familiar with)
The rules below are not formally stated until chapter 6, but knowing what you know about derivatives, you should be able to make the following connections:

<table>
<thead>
<tr>
<th>Integral Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\int a , dx = ax + C$, where $a$ is a constant</td>
</tr>
<tr>
<td>2. Power Rule for $x^n$ when $n \neq -1$: $\int x^n , dx = \frac{x^{n+1}}{n+1} + C$</td>
</tr>
<tr>
<td>3. Rule for $x^n$ when $n = -1$: $\int \frac{1}{x} , dx = \ln</td>
</tr>
<tr>
<td>4. $\int a^x , dx = \frac{a^x}{\ln a} + C$, where $a$ is a constant</td>
</tr>
<tr>
<td>5. $\int e^x , dx = e^x + C$</td>
</tr>
<tr>
<td>6. $\int \sin x , dx = -\cos x + C$</td>
</tr>
<tr>
<td>7. $\int \cos x , dx = \sin x + C$</td>
</tr>
<tr>
<td>8. $\int \sec^2 x , dx = \tan x + C$</td>
</tr>
<tr>
<td>9. $\int \csc^2 x , dx = -\cot x + C$</td>
</tr>
<tr>
<td>10. $\int \sec x \tan x , dx = \sec x + C$</td>
</tr>
<tr>
<td>11. $\int \csc x \cot x , dx = -\csc x + C$</td>
</tr>
<tr>
<td>12. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$</td>
</tr>
<tr>
<td>13. $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$</td>
</tr>
<tr>
<td>14. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$</td>
</tr>
</tbody>
</table>

**Example 3:** Use the evaluation part of the Fundamental Theorem of Calculus to evaluate each expression:

a) $\int_{0}^{3} x^2 \, dx$

b) $\int_{\frac{\pi}{2}}^{\pi} \cos x \, dx$

c) $\int_{-1}^{2} 3^x \, dx$

d) $\int_{4}^{9} f'(x) \, dx =$

This last example will be an extremely important concept as we go through the rest of the semester!

Using the evaluation part, we are going to develop the concept of the other part of the Fundamental Theorem of Calculus. Your book calls this *Part 1*, because it proves them in the opposite order. Our goal here isn’t really to prove the Fundamental Theorem of Calculus, Part 1, but to understand how it works.

We are going to determine how to take a derivative of a function that is defined as an integral and discuss what it means to define a function as an integral. Once we can do both of these things, we can answer all the same types of questions about increasing, decreasing, concave up, concave down, and inflection points that we did earlier in the year.

…you haven’t forgotten all those reasons have you?
Example 4: Suppose we wanted to find \( \int_a^x h'(t) \, dt \), where \( a \) is a constant.

Example 5: Now, suppose we wanted to find \( \frac{d}{dx} \left[ \int_a^x h'(t) \, dt \right] \), where \( a \) is a constant.

Notice how your answer is related to your original problem? …
This illustrates the “simple” version of the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus [Part #1 ... Simple]

\[
\frac{d}{dx} \left[ \int_a^x f(t) \, dt \right] = f(x)
\]

Example 6: If \( g(x) = \int_{-2}^x w(t) \, dt \), then \( g'(x) = ? \)

Example 7: \( \frac{d}{dx} \left[ \int_3^x 5t^2 - 6t + 1 \, dt \right] \)

The next example has limits on the integral that are functions of \( x \) … as opposed to simply a lower limit of \( x \) and an upper limit of a constant.

Example 8: \( \frac{d}{dx} \left[ \int_v^x h'(t) \, dt \right] \)

Notice how your answer is related to the original problem? …
This illustrates the “extended” version of the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus [Part 1 ... Extended]

\[
\frac{d}{dx} \left[ \int_v^x f(t) \, dt \right] = f(x) \cdot u'(x) - f(u(x)) \cdot v'(x)
\]

Example 9: Find \( \frac{d}{dx} \left[ \int_{t^2}^{3t} f(t) \, dt \right] \)

Example 10: Let \( g(x) = \int_{5x}^{3x^2} \sqrt{1 + t^2} \, dt \). Find \( g'(x) \).
Now that we have established how to take a derivative of an integral function, let’s look more closely at what it means to define a function using an integral.

**Example 11:** Let $g(x) = \int_{-2}^{x} w(t) \, dt$, where the graph of $w(t)$ is given below.

a) Find $g(0)$.

b) Find $g(2)$.

c) Find $g(-3)$.

Now to PUT IT ALL TOGETHER (and tie 1st and 2nd semester together) … Almost guaranteed to be on the AP exam!

**Example 16:** Suppose the function below is the graph of $f(t)$ and $g(x) = \int_{-1}^{x} f(t) \, dt$.

a) Complete the table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
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<tr>
<td>2</td>
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<tr>
<td>3</td>
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</tbody>
</table>

b) What are the intervals on which $g$ is increasing or decreasing? Justify each response.

c) What are the intervals on which $g$ is concave up or concave down? Justify each response.

d) For what value of $x$ does $g$ have a relative maximum? Justify your response.

e) For what value of $x$ does $g$ have an inflection point? Justify your response.