4.5 LINEARIZATION

Notecards from Section 4.5: Linearization; Differentials

Linearization

The goal of linearization is to approximate a curve with a line. Why? Because it's easier to use a line than a curve!
**All you have to do is find the equation of a tangent line and use the tangent line instead of the original function.

Example 1: Consider $f(x) = \sqrt{x}$. We all know that $f(4) = 2$, but without a calculator, how can we find $f(4.1)$?

a) Find the equation of the tangent line for $f(x)$ when $x = 4$. [Your book refers to this as $L(x)$.]

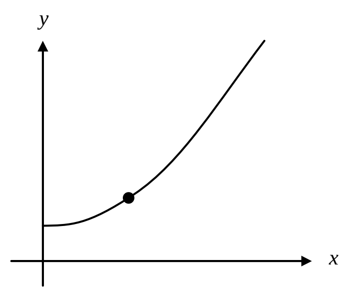
b) The tangent line you found is approximately the same as $f(x)$ "centered at $x = 4$".
Use your tangent line to approximate $f(4.1)$.

c) Use a calculator to approximate $f(4.1)$. How close is the approximation?

Differentials

Approximations aren't exact! (Aren't you glad you woke up this morning to hear that enlightening bit of information?!) If we use a tangent line to approximate a curve, it gives us a good estimate, as long as we don't go too far away from the center point. Wouldn't it be nice if we knew how far off our approximation is going to be?

Example 2: Consider the function f pictured to the right.



a) Label the center point $(c, f(c))$, and draw the tangent line at that point.

b) What is the equation of the *tangent line* you drew?
Keep in mind … this is just a linearization of the curve.

c) Move Δx to the right of c. What is the y-value of this new point on the tangent line?

d) The ACTUAL change in y, Δy , is given by ___________________.

When Δx is very small (infinitesimally small), we say that $\Delta x = dx$ (the differential of x).

e) We will denote the APPROXIMATE change in y with dy. Find dy.

What differentials allow us to do is treat the *dy* and *dx* terms of the notation $\frac{dy}{dx}$ as separate terms.

Finding a differential is simply finding a derivative using different notation. JUST REMEMBER THAT …
dy is a small change in *y*, and *dx* is a small change in *x*.

Example 3: Suppose $y = 5x^3 - 3x^2$. Find the derivative using differentials.

Example 4: Find the differential *dy* when *dx* = 0.01 and *x* = 2, if $y = x^5 - 4x^3$. *Explain* what you've found.

"Errors in an approximation" … (i.e. how "close" was the approximation)
(Actual Change in Function) – (Approximate Change in Function) = $\Delta y - dy$

Example 5: Use differentials to estimate $\sqrt[3]{29}$. Find the error in your approximation.

Example 6: The measurement of a side of a square is found to be 15 centimeters. The possible error in measuring the side is 0.05 centimeter. Find the approximate error in computing the area of the square.

4.6 RELATED RATES

When one or more values in an equation change over time, we have related rates. We simply write equations that you might have written prior to this course (with no motion taking place), then differentiate them with respect to time, *t*.

Example 1: Do you remember how we found the derivative of $x^2 + y^2 = 9$?

This derivative was $\frac{dy}{dx}$ … the derivative of *y* with respect to *x*.

Example 2: How do you suppose we take the derivative of $a^2 + b^2 = c^2$ with respect to *t*.

In related rates problems, you will be asked to solve for one of these rates. In order to do that, you will have to have enough given information to find values for all other variables in the problem.

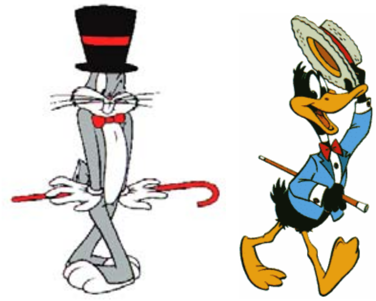
Related Rates problems typically fall into 3 categories … Pythagorean Theorem, Right-Triangle Trigonometry, and known or given formulas involving Volumes, Areas, or other scenarios.

Your answer needs to be written using correct ___________.

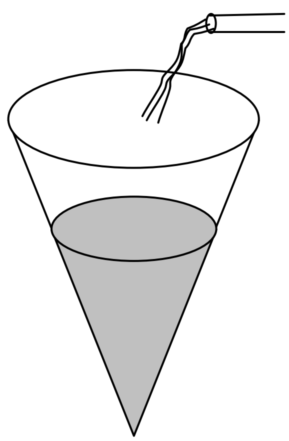
Example 3: If *h* is measured in cm and *t* is measured in min, then $\frac{dh}{dt}$ is measured is _____________.

Example 4: Tweety is resting in a bird house 24 feet off the ground. Using a 26 foot ladder which he leaned against the pole holding the bird house, Sylvester tries to steal the small yellow bird. Tweety's bodyguard, Hector the dog, starts pulling the base of the ladder away from the pole at a rate of 2 ft/s. How fast is the ladder falling when it is 10 feet off the ground?

Example 5: Bugs and Daffy finished their final act on the *Bugs and Daffy Show* by dancing off the stage with a spotlight covering their every move. If they are moving off the stage along a straight path at a speed of 4 ft/s, and the spotlight is 20 ft away from this path, what rate is the spotlight rotating when they are 15 feet from the point on the path closest to the spotlight?



Example 6: A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep. The volume of a circular cone with radius *r* and height *h* is given by $V = \frac{1}{3}\pi r^2 h$.



In a related rates problem, you have an equation relating two or more things that ***change over time***, and we want to find the rate of change (a derivative) of one of these things. It is important to understand that without some conditions given to use, we cannot solve the problem.

For those of you who like some step-by-step procedures …

Guidelines for Solving Related-Rate Problems

Step 1: Read the problem, really! You'd be amazed how many people skip this step. Then read it again! ☺

Step 2: Draw a diagram showing what's going on. Identify all relevant information and assign variables to what's changing. Use the general case (numbers for values that NEVER change in this situation, and variables for anything that is changing).

♬ : Related Rates usually involve motion … any diagram you draw is like a still picture of what is occurring. Any part of your picture that NEVER changes can be labeled with a constant (or number), but any part of your picture that is in motion or is changing MUST be labeled with a variable!

In other words, if the radius of a circle is increasing and you are asked to find the rate of change in the area at the exact moment when the radius is 5 cm, then your diagram would be a circle, but you would NOT label the radius 5 because it is changing … you would label the radius *r*.

Step 3: Find the equation that gives the relationship between the variables you just named in step 2. This is sometimes the hardest part, but most problems fall into three categories … a triangle that you can use a trigonometric ratio (involving sides and angles), the Pythagorean theorem (involving all 3 sides of a right triangle), or a known formula like Area, Volume, Distance, etc.

Step 4: Find the particular information (values of variables at the exact moment you drew your diagram) for the problem and write it down. Then, list what you are looking for (normally this would be a derivative).

Step 5: Implicitly differentiate the equation with respect to time *t*. Usually this equation will have at least two derivatives. If it has more than two, be sure you have enough information, *or* you may have to find a relationship between two of the variables, and rewrite the equation in step 3 using this relationship.

Step 6: Plug in the particular information, and solve for the desired quantity. DO NOT DO THIS UNTIL AFTER YOU HAVE TAKEN THE DERIVATIVE!

Step 7: Write down your answer and circle it with your favorite color. (be sure to use correct units)