4.5 LINEARIZATION

Notecards from Section 4.5: Linearization; Differentials; Absolute vs Relative vs Percentage Change

Linearization

The goal of linearization is to approximate a curve with a line. Why? Because it's easier to use a line than a curve!
**All you have to do is find the equation of a tangent line and use the tangent line instead of the original function.

Example 1: Consider $f(x)=\sqrt{x}$. We all know that $f(4) = 2$, but without a calculator, how can we find $f(4.1)$?

a) Find the equation of the tangent line for $f(x)$ when $x = 4$. [Your book refers to this as $L(x)$.]

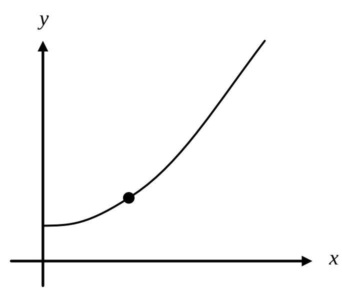
b) The tangent line you found is approximately the same as $f(x)$ "centered at $x = 4$".
Use your tangent line to approximate $f(4.1)$.

c) Use a calculator to approximate $f(4.1)$. How close is the approximation?

Differentials

Approximations aren't exact! (Aren't you glad you woke up this morning to hear that enlightening bit of information?!) If we use a tangent line to approximate a curve, it gives us a good estimate, as long as we don't go too far away from the center point. Wouldn't it be nice if we knew how far off our approximation is going to be?

Example 2: Consider the function f pictured to the right.



a) Label the center point $(c, f(c))$, and draw the tangent line at that point.

b) What is the equation of the *tangent line* you drew?
Keep in mind … this is just a linearization of the curve.

c) Move Δx to the right of c. What is the y-value of this new point on the tangent line?

d) The ACTUAL change in y, Δy, is given by ___________________.

When Δx is very small (infinitesimally small), we say that $\Delta x = dx$ (the differential of x).

e) We will denote the APPROXIMATE change in y with dy. Find dy.

What differentials allow us to do is treat the *dy* and *dx* terms of the notation $\frac{dy}{dx}$ as separate terms.

Finding a differential is simply finding a derivative using different notation. JUST REMEMBER THAT …
dy is a small change in *y*, and *dx* is a small change in *x*.

Example 3: Find the differential *dy* when *dx* = 0.01 and *x* = 2, if $y = x^5 - 4x^3$. *Explain* what you've found.

Percentages & Errors

When you have been asked to find a percentage change, you always find the "change" divided by the original amount.

"Approximate percent change in *y*" can be found by $\frac{dy}{y}$, since *dy* is the approximate change in *y*.

When you make a mistake in a measurement and then use that measurement to calculate something else, the resulting effect the original error has on the calculation can be found using differentials. The "errors" will be the "change".

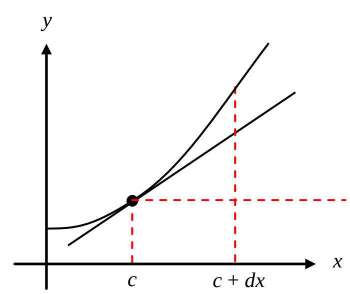
"Approximate errors" will be found using differentials
dx = approximate "error" in the measurement of *x*
dy = approximate "error" in the measurement of *y*

"Actual errors" are found by finding the change in the function values (Δy).

$$f(c+dx) - f(c)$$

"Errors in an approximation" … (i.e. how "close" was the approximation)
(Actual Change in Function) – (Approximate Change in Function)

$$\Delta y - dy$$



Example 4: Use differentials to estimate $\sqrt[3]{29}$. Find the error in your approximation.

Example 5: The measurement of a side of a square is found to be 15 centimeters. The possible error in measuring the side is 0.05 centimeter.

a) Find the approximate error in computing the area of the square.

b) Approximate the percent error in computing the area of the square.

c) Estimate the maximum allowable percent error in measuring the side if the error in computing the area cannot exceed 2.5%

4.6 RELATED RATES

When one or more values in n equation change over time, we have related rates. We simply write equations that you might have written prior to this course (with no motion taking place), then differentiate them with respect to time, *t*.

Example 1: Do you remember how we found the derivative of $x^2 + y^2 = 9$?

This derivative was $\frac{dy}{dx}$ … the derivative of *y* with respect to *x*.

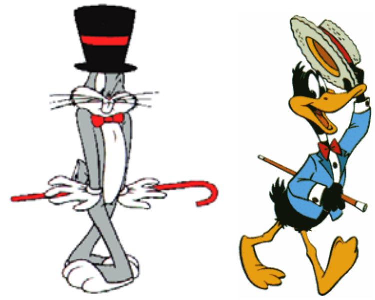
Example 2: How do you suppose we take the derivative of $a^2 + b^2 = c^2$ with respect to *t*.

In related rates problems, you will be asked to solve for one of these rates. In order to do that, you will have to have enough given information to find values for all other variables in the problem.

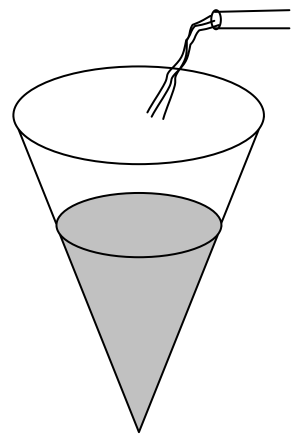
Example 3: Suppose you are told that the particle moving along the curve $y = x^2$ is moving horizontally at 2 cm/s. Find the rate of change in the particle's vertical position at the exact moment the particle is at (3, 9).

Example 4: Tweety is resting in a bird house 24 feet off the ground. Using a 26 foot ladder which he leaned against the pole holding the bird house, Sylvester tries to steal the small yellow bird. Tweety's bodyguard, Hector the dog, starts pulling the base of the ladder away from the pole at a rate of 2 ft/s. How fast is the ladder falling when it is 10 feet off the ground?

Example 5: Bugs and Daffy finished their final act on the *Bugs and Daffy Show* by dancing off the stage with a spotlight covering their every move. If they are moving off the stage along a straight path at a speed of 4 ft/s, and the spotlight is 20 ft away from this path, what rate is the spotlight rotating when they are 15 feet from the point on the path closest to the spotlight?



Example 6: A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep. The volume of a circular cone with radius r and height h is given by $V=\frac{1}{3}\pi r^2 h$.



In a related rates problem, you have an equation relating two or more things that *change over time*, and we want to find the rate of change (a derivative) of one of these things. It is important to understand that without some conditions given to use, we cannot solve the problem.

Guidelines for Solving Related-Rate Problems

Step 1: Read the problem, really! You'd be amazed how many people skip this step. Then read it again! ☺

Step 2: Draw a diagram showing what's going on. Identify all relevant information and assign variables to what's changing. Use the general case (numbers for values that NEVER change in this situation, and variables for anything that is changing).

♬: Related Rates usually involve motion … any diagram you draw is like a still picture of what is occurring. Any part of your picture that NEVER changes can be labeled with a constant (or number), but any part of your picture that is in motion or is changing MUST be labeled with a variable!

In other words, if the radius of a circle is increasing and you are asked to find the rate of change in the area at the exact moment when the radius is 5 cm, then your diagram would be a circle, but you would NOT label the radius 5 because it is changing … you would label the radius r.

Step 3: Find the equation that gives the relationship between the variables you just named in step 2. This is sometimes the hardest part, but most problems fall into three categories … a triangle that you can use a trigonometric ratio (involving sides and angles), the Pythagorean theorem (involving all 3 sides of a right triangle), or a known formula like Area, Volume, Distance, etc.

Step 4: Find the particular information (values of variables at the exact moment you drew your diagram) for the problem and write it down. Then, list what you are looking for (normally this would be a derivative).

Step 5: Implicitly differentiate the equation with respect to time t. Usually this equation will have at least two derivatives. If it has more than two, be sure you have enough information, *or* you may have to find a relationship between two of the variables, and rewrite the equation in step 3 using this relationship.

Step 6: Plug in the particular information, and solve for the desired quantity. DO NOT DO THIS UNTIL AFTER YOU HAVE TAKEN THE DERIVATIVE!

Step 7: Write down your answer and circle it with your favorite color. (be sure to use correct units)

5.1 ESTIMATING WITH FINITE SUMS

Notecards from Section 5.1 & 5.5: LRAM, RRAM, MRAM, Trapezoidal Rule

Example 1: Suppose from the 2nd to 4th hour of your road trip, you travel with the cruise control set to exactly 70 miles per hour for that two hour stretch. How far have you traveled during this time?

Example 2: Sketch a graph modeling the situation in the above example. Geometrically, how can we indicate the total distance traveled?

Example 3: What if the velocity was NOT constant. Say, for instance the velocity in miles per hour is given by the function $v(t) = 10t - t^2$, where *t* is in hours, and we wanted to know the total distance traveled during the first 10 hours. Sketch this graph below. Geometrically speaking, do you think we can find the total distance traveled in the same was as before? Why or why not?

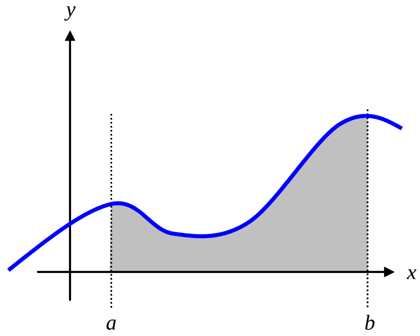
Word of caution to those brave few who are actually reading this … The following paragraphs are extremely important to the conceptual understanding of what we are about to do in Calculus. However, since you really haven't done anything yet, it might make you a little dizzy at first, so come back and read it again later. If you're reading this for the first time, you might experience that same feeling you get when you've been on the Tilt –A – Whirl one too many times at the fair. (Never been on the Tilt – A – Whirl? … well, take my word for it, it's not something you want to ride 10 minutes after eating a corn dog and a funnel cake!) Well, I warned you, but you've kept on reading anyway, so here it goes …

The key to finding the total distance traveled in the last example in a method similar to the first example is to break the time intervals into such short segments, that the velocity over those time segments is almost constant (this will require quite a few intervals). If the velocity is almost constant for each time interval, then we can find the distance traveled for each time interval (which is just the area of an extremely thin rectangle) and add all the areas of all the rectangles together. Sounds simple enough, right? Can you guess what extremely important calculus concept is involved?

We will spend MUCH more time with this later, but it turns out that if we are given the graph of a rate of change (like velocity in miles per hour) we will be able to find the total accumulated change over an interval (like total distance traveled, in miles) by finding the area under the curve.

OK, that last paragraph or two may not have made perfect sense to you … YET! … For now, THE GOAL is to figure out a way to find the area under the curve. This chapter actually discusses 5 ways to approximate this area, LRAM, RRAM, MRAM, Trapezoid Rule, and Simpson’s Rule, but we're only going to deal with 4 of them. (Simpson’s Rule is not in the AP curriculum)

The Area Problem and the Rectangular Approximation Method (RAM)
(a.k.a. Riemann Sums)



Suppose we wanted to know the area of the region bounded by a curve, the x – axis, and the lines $x = a$ and $x = b$, as shown at the right.

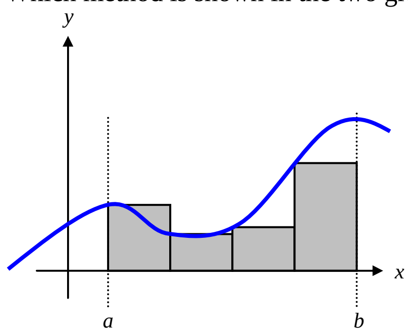
The first step is to divide the interval from a to b into subintervals. (The examples below show 4 and 8 subintervals, respectively.)

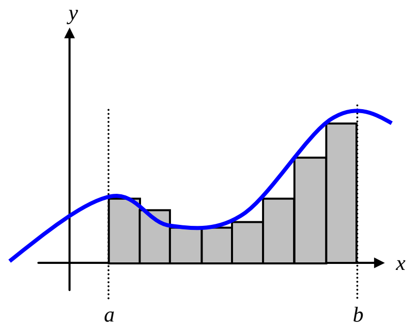
After dividing the given interval into subintervals, we can then draw rectangles using the width of each subinterval as the base.

The *height of each rectangle is determined by the function value at a point in the specific subinterval*, and can be determined using 3 different methods.

We could use the left endpoint of each subinterval (called LRAM), the right endpoint of each subinterval (RRAM), or the midpoint of each subinterval (MRAM).

Example 4: Which method is shown in the two graphs below?





Example 5: The total area under the curve then is approximately equal to the total area of all the rectangles. Which of the graphs above gives a better approximation of the area under the curve? Why? How could it be further improved?

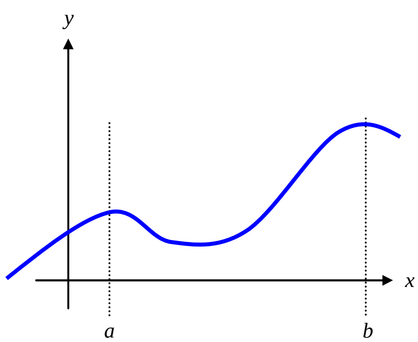
Summary of the Process: A sketch is almost mandatory!

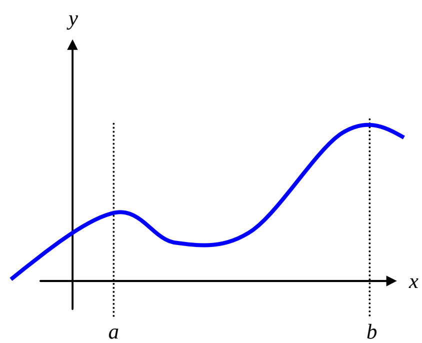
Step 1: Divide (or Partition) the interval into n subintervals.

Step 2: Create n rectangles whose base equals the width of each subinterval and whose height is determined by the function value at the left endpoint, the right endpoint, or the midpoint of the subinterval.

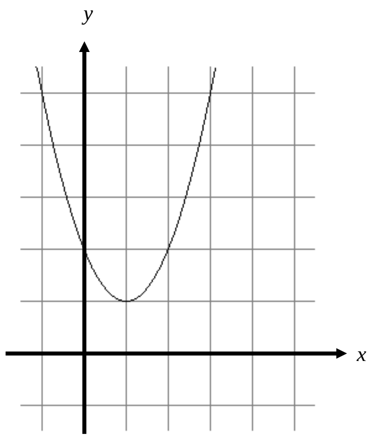
Step 3: Find the area of all n rectangles and add them together.

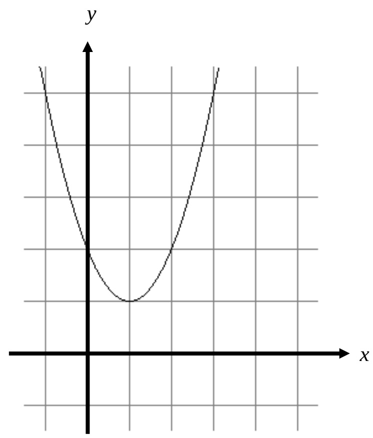
Example 6: Illustrate the use of RRAM and MRAM on the graphs below. (use 4 rectangles)

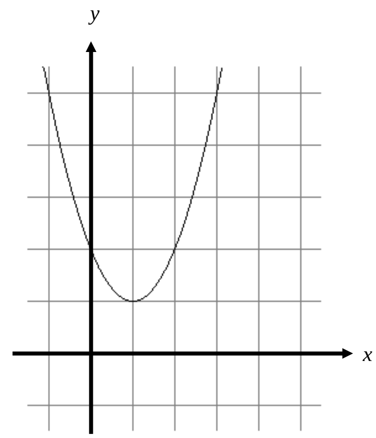




Example 7: Use 4 rectangles to approximate the area under the graph of $y = x^2 - 2x + 2$ from $x = 1$ to $x = 3$.
Use LRAM, RRAM, and then MRAM.







Example 8: Using your rectangles as a guide, find each approximation.

a) LRAM =

b) RRAM =

c) MRAM =

Example 9: It is not necessary to have a graph to estimate the area. Suppose the table below shows the velocity of a model train engine moving along a track for 10 seconds.

Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

a) Using a left Riemann Sum with 10 subintervals, estimate the distance traveled by the engine in the first 10 seconds.

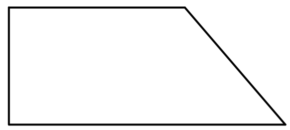
b) Using a Midpoint Riemann Sum with 5 subintervals, estimate the distance traveled by the engine in the first 10 seconds.

The Trapezoidal Rule (Really §5.5)

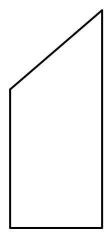
While rectangles make a fairly good approximation, it's easy to see that we're going to need a lot of them to provide a good estimate. We can find a better estimate in less time if we use trapezoids. If we were to partition the interval into subintervals like we did before, we can use each subinterval to create a trapezoid if we just connect the function values of the left and right endpoints. Before we begin, let's make sure you understand the area formula for a trapezoid.

Area of a Trapezoid: $A = \frac{1}{2} \cdot h \cdot (b_1 + b_2)$

While not all trapezoids must look like this, the one's we're going to be using will, so we'll stick with this picture. Label all the parts of the area formula on the picture below.

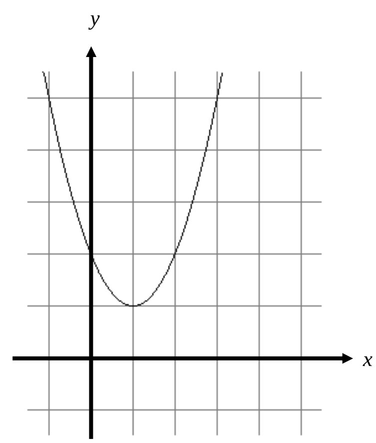


The biggest difference will be the orientation of the trapezoid. The ones we are going to be drawing will look like



Draw a set of axes on the picture above and a function that goes through the top left and top right points of the trapezoid. The "height" of the trapezoid is just the width of a subinterval, and the "bases" are going to be the function values of the left and right endpoints.

Example 10: Use 4 trapezoids to approximate the area under the curve $y = x^2 - 2x + 2$ from $x = 1$ to $x = 3$. Sketch the trapezoids.



By the way … The trapezoid rule connects the left and right hand endpoints with a segment. This method of approximation turns out to be a pretty good, but if you were to connect the endpoints with a curve (namely a parabola) the approximation would be even better. Connecting the endpoints with a parabola and finding the area of the resulting shape is the basis behind the fifth method of approximation called Simpson's Rule. You can read about it in your book if you find yourself just dying of curiousity, but it's not in the AP curriculum.

5.2 DEFINITE INTEGRALS

Notecards from Section 5.2: Riemann Sum; Definition of Definite Integral; Properties of a Definite Integral (more in the next section)

Riemann Sums

In the last section we found the area under a curve by finding the area of a finite number of rectangles (LRAM, RRAM, and MRAM) or a finite number of trapezoids (TRAPEZOID RULE). Every one of these was an example of what is called a **Riemann Sum**. We're going to stick with RECTANGLES for the time being.

The following steps illustrate what has to happen in order for the sum to be considered a RIEMANN Sum.

Step 1: Start with a *continuous* function on a *closed* interval.

Step 2: ***Partition*** the interval into n subintervals. The k^{th} subinterval has width $\triangle x_k$.
(We made sure they were all the same size, but it turns out, it really doesn't matter for what we're doing here.)

Step 3: In each subinterval, pick ***any*** number and call the number picked from the kth subinterval c_k
(LRAM picked the left endpoint … RRAM picked the right endpoint … MRAM picked the midpoint)

Step 4: For each interval, using the width, $\triangle x_k$, of the interval as the base, create a rectangle that extends from the x – axis to the function value, $f(c_k)$, of the number you picked in each interval.
(♬: Some of these rectangles could lie below the x – axis.)

Step 5: On each interval, form the product $f(c_k)\cdot\triangle x_k$.
(If all our rectangles lie above the x-axis, this would have been the area of each rectangle)

Step 6: Find the SUM of each of these products.

$$f(c_1)\cdot\triangle x_1 + f(c_2)\cdot\triangle x_2 + f(c_3)\cdot\triangle x_3 + f(c_4)\cdot\triangle x_4 + \cdots + f(c_n)\cdot\triangle x_n = \sum_{k=1}^{n} f(c_k)\cdot\triangle x_k .$$

Following these steps gives you a **Riemann Sum for *f* on the interval [*a*, *b*]**. Every Riemann sum depends on the partition you choose (i.e. the number of subintervals) and your choice of the number within each interval, c_k.

Definite Integral as a Limit of a Riemann Sum

Our Goal … Develop a Mathematical DEFINITION of a Definite Integral … (i.e. What is a Definite Integral?)

While "AREA" is inherently positive, a Riemann sum can be negative if the rectangles lie below the x-axis.

A DEFINITE INTEGRAL is defined as a Limit of a Riemann Sum.

Option #1: If you noticed in step 2 above, we didn't care if our subintervals were the same width. If we use the notation $\|P\|$ to denote the longest subinterval length we can force the longest subinterval length to 0 using a limit of the Riemann Sum as follows:

$$\lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k$$

Option #2: If we make sure the subintervals are all the same width, we can increase the number of rectangles to infinity using a limit of the Riemann Sum as follows:

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\cdot\Delta x$$

Notation for Definite Integrals

The limit notation we used last is the form we will use to develop Integral notation. As the number of rectangles goes to infinity, the width of each rectangle, Δx, goes to zero. As we did in the section on differentials, we are going to use the notation *dx* to represent this infinitely tiny distance.

The summation notation of sigma is going to be replaced with an *Integral Sign* , $\int$, which looks somewhat like a giant "S" for sum.

The $f(c_k)$ which represented a different function value for each interval is going to be replaced with $f(x)$ since the x – values are going to be soooooooo close together it's almost as if we are evaluating the function at EVERY x – value in the interval $[a, b]$. Combining all of this we have the following notation:

$$\int_a^b f(x)\,dx$$

We read the notation above as "***<u>The Integral of f of x from a to b</u>***"

Important ♬ (Actually it's a Theorem): IF the function is continuous , THEN the Definite Integral will exist. However, the converse, while true some of the time is NOT ALWAYS true.

Using Definite Integrals as Area

We can define the **area under the curve *y* = *f* (*x*) from *a* to *b*** as an *integral* from *a* to *b* …
… AS LONG AS THE CURVE IS NONNEGATIVE AND INTEGRABLE on the closed interval $[a, b]$.

Drawing a picture and using geometry is still a valid method of finding areas in this class!

Example 1: For each of the following examples, sketch a graph of the function, shade the area you are trying to find, then use geometric formulas to evaluate each integral.

a) $\int_2^9 3\,dx$

b) $\int_{-2}^1 |x|\,dx$.

c) $\int_{-3}^3 \sqrt{9-x^2}\,dx$

So … what happens if the "area" is below the *x*-axis … as I mentioned before, "area" is inherently positive, but a Riemann sum … and therefore an Integral can have negative values if the curve lies below the *x*-axis.

Example 2: Consider the function $f(x) = 3 - x$. Sketch a graph of this function.

a) What is the "AREA" between the curve and the x – axis between $x = 4$ and $x = 8$?

b) Evaluate $\int_4^8 (3 - x)\, dx$

Example 3: Given $\int_0^{\pi} \sin x\, dx = 2$, use what you know about a sine function to evaluate the following integrals.

a) $\int_{\pi}^{2\pi} \sin x\, dx$

b) $\int_0^{2\pi} \sin x\, dx$

c) $\int_0^{\pi/2} \sin x\, dx$

d) $\int_{-\pi}^{\pi} \sin x\, dx$

e) $\int_0^{\pi} (2 + \sin x)\, dx$

The FnInt Function of your TI – 83+

By this point, hopefully you understand the following concepts:

1. The limit of a Riemann Sum is used to define a Definite Integral
2. A Definite Integral can be used to find the Area under a curve if the curve is above the *x* – axis, and if the curve is below the *x* – axis the value of the definite integral is "negative area" … even though no one in their right mind would ever actually use that phrase in a math class if they wanted to be taken seriously!
3. Since the Definite Integral can be thought of as Area, you can draw a picture and use geometric formulas to find the areas.

BUT … what happens if you were asked right now, this instant, today to find the definite integral of a function that doesn't lend itself to nice geometric shapes?

The good news for now, is you don't even have to worry about how to do these by hand … YET!
You get to use your calculator! … At least until the next lesson ☺

<u>How to Use Your Calculator to Find a Definite Integral:</u>

The syntax for using your calculator is as follows: fnInt(*function*, *x*, *lower bound*, *upper bound*)

1. Press MATH
2. Press 9: fnInt(
3. Follow the syntax above
   … enter the function, *x*, lower bound, upper bound (be sure to enter a comma between each)

Example 4: Evaluate $\int_1^3 \left(x^2 - 2x + 2\right) dx$

Example 5: Evaluate $3 + 2\int_0^{\pi/3} \tan x \, dx$

You can also do the same thing from the graphing screen.

Example 6: Graph $y = \sqrt{x}$ on a standard viewing window. Evaluate $\int_1^8 \sqrt{x}\, dx$.

Press 2^{nd} TRACE (which is CALC), 7: ∫ f(x)dx, enter Lower Bound as 1, enter Upper Bound as 8.

… the down side to using this method is that you MUST be able to set your window to SEE everything.

5.3 DEFINITE INTEGRALS AND ANTIDERIVATIVES

Notecards from Section 5.3: Properties of a Definite Integral; Average Value of *f*(*x*); Relationship between Average Value of *f*(*x*) and Average Rate of Change

In the last section we defined the definite integral as a limit of a Riemann Sum, thus we can use the properties of limits to develop properties of the definite integral. The proofs of each of the rules below are derived directly from the properties of limits and Riemann Sums.

Rules for Definite Integrals

1. Order of Integration: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

If you reverse the *order* of integration you get the opposite answer.

2. Zero: $\int_a^a f(x)\,dx = 0$

This should make sense if you think about the "area" of a rectangle with no width.

3. Constant Multiple: If k is any constant, then $\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx$

Taking the constant out of the integral many times makes it simpler to integrate.

4. Sum and Difference: $\int_a^b \left[f(x)\pm g(x)\right]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

This allows you to integrate functions that are added or subtracted separately. Notice, there are NO rules here for two functions that are multiplied or divided … that comes later!

5. Additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Pay close attention to the limits of integration … this comes in handy when dealing with total area or other functions where we need to break them into smaller parts.

Example 1: Given $\int_2^6 f(x)\,dx = 10$ and $\int_2^6 g(x)\,dx = -2$, find the following:

a) $\int_2^6 \left[f(x)+g(x)\right]dx$

b) $\int_2^6 \left[g(x)-f(x)\right]dx$

c) $\int_2^6 3f(x)\,dx$

d) $\int_2^6 \left(f(x)+2\right)dx$

Example 2: Given $\int_0^5 f(x)\,dx = 10$ and $\int_5^7 f(x)\,dx = 3$, find the following:

a) $\int_0^7 f(x)\,dx$

b) $\int_5^0 f(x)\,dx$

c) $\int_5^5 f(x)\,dx$

d) $\int_0^5 3f(x)\,dx$

Average Value of a Function

Suppose you wanted to find the average temperature during a 24 hour period. How could you do it?

Suppose $f(t)$ represents the temperature at time t, measured in hours since midnight. One way to start is to measure the temperature at n equally spaced times $t_1, t_2, t_3, \ldots t_n$ and then average those temperatures.

Example 3: Using this method, write an expression for the AVERAGE temperature.

The larger the number of measurements, the more accurately this will reflect the average temperature. Notice we can write this expression as a Riemann sum by first noting that the interval between measurements will be $\Delta t = \frac{24}{n}$, so $n = \frac{24}{\Delta t}$.

Example 4: Substitute this value of n into your expression above and simplify.

Example 5: The last expression gives us an approximate Average Temperature. As $n \to \infty$ (meaning we are taking a lot of temperature readings) this Riemann Sum becomes a definite integral. Write the Definite Integral that gives us the Average Temperature since midnight.

Example 6: Do you think that there is any point during the day that the temperature reading on the thermometer is the *exact* value of the average temperature?

The process that we just used to find the average temperature is used to find the Average Value of any function.

The Average Value of a Function

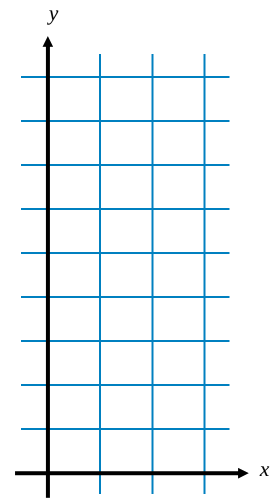
If f is integrable on $[a, b]$, its **average value** on $[a, b]$ is given by

$$\text{AVERAGE VALUE} = \frac{1}{b-a}\int_a^b f(x)\,dx \;\ldots\text{ or }\ldots\; \text{AVERAGE VALUE} = \frac{\int_a^b f(x)\,dx}{b-a}$$

The *average value of a function* is just … "the integral over the interval".

To get a more geometric idea of what the average value is, complete the following examples:

Example 7: Graph the function $y = x^2$ on [0,3] on the grid to the right.



Example 8: Set up a definite integral to find the average value of y on [0, 3], then use your calculator to evaluate the definite integral.

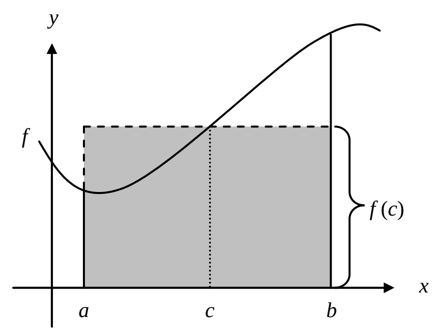
Example 9: Graph this as a value as a function on the grid to the right. Does the function ever actually equal this value? If so, at what point(s) in the interval does the function assume its average value?

Example 10: What do you suppose is the relationship between the area between the x – axis and the curve $y = x^2$ on [0, 3] and the area of the rectangle formed using the average value as the height and the interval [0, 3] as the width?

<u>*The Mean Value Theorem for Definite Integrals … see the connection … Mean … Average …*</u>

The Mean Value Theorem for Integrals basically says that if you are finding the area under a curve between $x = a$ and $x = b$, then there is *some* number c between a and b whose function value you can use to form a rectangle that has an area equal to the area under the curve.

Example 11: What is an expression that could be used to determine the *area under the curve* from a to b ?



Example 12: What is the area of the shaded rectangle?

This value of $f(c)$ is just the *Average Value* of f on the interval $[a, b]$.

So another way to look at this is the Mean Value Theorem for Integrals just says that at some point within the interval the function MUST equal its average value.

Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

…Once again, we have a theorem that tells us a value of c exists, but the theorem doesn't actually find it for us!

It is greatly important that you understand the difference between *average rate of change* and *average value*.

More on this after we finish §5.4. For now, understand that the average rate of change is simply the "slope between two points" on a given function and the average value of the function is the "integral divided by the interval".

5.4 FUNDAMENTAL THEOREM OF CALCULUS

Notecards from Section 5.4: Fundamental Theorem of Calculus Part 1 (Extended); Fundamental Theorem of Calculus Part 2 (Evaluation)

Do you remember the Fundamental Theorem of Algebra? Just thought I'd ask …

The Fundamental Theorem of Calculus has two parts. These two parts tie together the concept of integration and differentiation and is regarded by some to by the most important computational discovery in the history of mathematics! Be honest! … Deep down, you're not only really impressed, but you're filled with anticipation! I know it's hard, but try to contain your excitement!

Example 1: In section 5.2 we found an estimate for the distance traveled by finding the area between a velocity function and the x-axis. If $v\,(t)$ is that velocity function (that is above the x-axis) and the time is from 0 to 10 seconds, how might we use calculus to define the distance traveled?

Example 2: Suppose a car's position is given by $s(t)=\frac{3}{2}t^2+30t+25$ where t is time in seconds, and $0\le t\le 10$.

a) What is the position of the car at $t = 0$ seconds?

b) What is the position of the car at $t = 10$ seconds?

c) What is the change in position of the car from time $t = 0$ to time $t = 10$ seconds?

d) How does this question relate to the previous example?

The Fundamental Theorem of Calculus [The Evaluation Part]

If f is continuous at every point of $[a, b]$,

$$\int_a^b f(x)\,dx = F(b)-F(a),$$

<u>where F(x) is an antiderivative of f (x).</u>

(… that last phrase is actually the toughest part! … below is a list of antiderivatives you should become familiar with)

The rules below are not formally stated until chapter 6, but knowing what you know about derivatives, you should be able to make the following connections:

Integral Formulas

1. Power Rule for x^n when $n \neq -1$: $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$
2. Rule for x^n when $n = -1$: $\int \frac{1}{x} \, dx = \ln|x| + C$
3. $\int e^{kx} \, dx = \frac{e^{kx}}{k} + C$
4. $\int \sin(kx) \, dx = \frac{-\cos(kx)}{k} + C$
5. $\int \cos(kx) \, dx = \frac{\sin(kx)}{k} + C$
6. $\int \sec^2(x) \, dx = \tan(x) + C$
7. $\int \csc^2(x) \, dx = -\cot(x) + C$
8. $\int \sec(x)\tan(x) \, dx = \sec(x) + C$
9. $\int \csc(x)\cot(x) \, dx = -\csc(x) + C$

Example 3: $\int_0^3 x^2 \, dx$

Example 4: $\int_{\pi/2}^{\pi} (1 + \cos x) dx$

Example 5: $\int_{-1}^{2} 3^x \, dx$

Example 6: $\int_4^9 f'(x) dx =$

This last example will be an extremely important concept as we go through the rest of the semester!

Using the evaluation part, we are going to develop the concept of the other part of the Fundamental Theorem of Calculus. You're book calls this *Part 1*, because it proves them in the opposite order. Our goal here isn't really to prove the Fundamental Theorem of Calculus, Part 1, but to understand how it works.

First, a quick overview …

#1. We are going to create a function that is defined as an integral … then,

#2: Using this function we are going to find the derivative of this function … thus tying the two concepts of calculus together forever!!!

Keep in mind that if we can define a function as an integral and take a derivative, then we can answer all the same types of questions about increasing, decreasing, concave up, concave down, and inflection points that we did earlier in the year. …you haven't forgotten all those reasons have you?

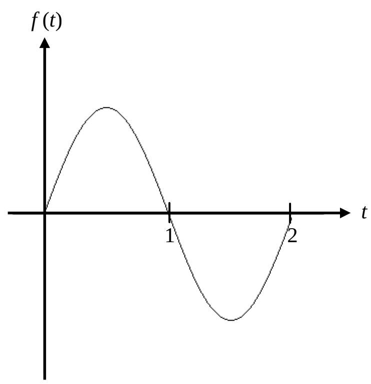
Step #1: So, to see how it is possible to define a function using an integral, consider the examples below.

The graph of $f(t)$ given below has odd symmetry and is periodic (with period = 2). Also, $\int_0^1 f(t)\,dt = \frac{4}{3}$.

Example 7: Let $F(x) = \int_0^x f(t)\,dt$.

a) Complete the following table:

x	$F(x)$
–1	
0	
1	
2	
3	

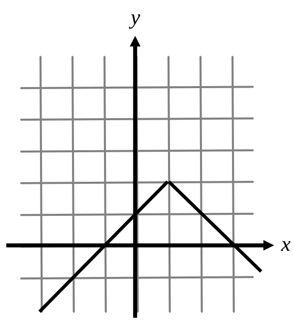


Example 8: Let $g(x) = \int_{-2}^x w(t)\,dt$, where the graph of $w(t)$ is given below.

a) Find $g(0)$.

b) Find $g(2)$.

c) Find $g(-3)$.



OK … now that we've established that we are able to define a function as an integral, let's talk about how to find a derivative of such a function. While interpreting a function defined as an integral is a valid skill in its own right, our goal here is to take a derivative of that function and to see how it relates to what we started with.

Earlier (example 6), we found $\int_{4}^{9} f'(x)\,dx =$ … remember how we did this?

Example 9: Suppose we wanted to find $\int_{a}^{x} h'(t)\,dt$, where *a* is a constant.

Example 10: Now, suppose we wanted to find $\frac{d}{dx}\left[\int_{a}^{x} h'(t)\,dt\right]$, where *a* is a constant.

Notice how your answer is related to your original problem? …
This illustrates the "simple" version of the Fundamental Theorem of Calculus.

> *The Fundamental Theorem of Calculus [Part #1 … Simple]*
>
> $$\frac{d}{dx}\left[\int_{a}^{x} f(t)\,dt\right] = f(x)$$

Example 11: If $g(x) = \int_{-2}^{x} w(t)\,dt$, then $g'(x) =$?

Example 12: $\frac{d}{dx}\left[\int_{3}^{x} \left(5t^2 - 6t + 1\right) dt\right]$

The next example has limits on the integral that are functions of *x* … as opposed to simply a lower limit of *x* and an upper limit of a constant.

Example 13: $\frac{d}{dx}\left[\int_{v(x)}^{u(x)} h'(t)\,dt\right]$

Notice how your answer is related to the original problem? …
This illustrates the "extended" version of the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus [Part 1 … Extended]

$$\frac{d}{dx}\left[\int_{v(x)}^{u(x)} f(t)\,dt\right] = f(u(x))\cdot u'(x) - f(v(x))\cdot v'(x)$$

Example 14: Find $\frac{d}{dx}\left[\int_{x^2}^{3x} f(t)\,dt\right]$

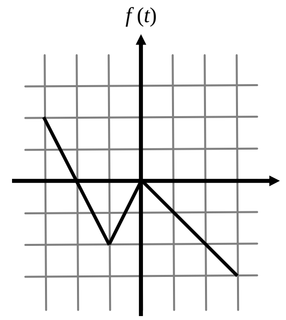
Example 15: Suppose $g(x) = \int_{5x}^{3x^2} \sqrt{1+t^3}\,dt$. Find $g'(x)$.

Putting it all together …

Example 16: Suppose the function below is the graph of $f(t)$ and $g(x)=\int_{-1}^{x} f(t)\,dt$.

a) Complete the table:

x	$g(x)$
–3	
–2	
–1	
0	
1	
2	
3	



b) What are the intervals on which g is increasing or decreasing? Justify each response.

c) What are the intervals on which g is concave up or concave down? Justify each response.

d) For what value of x does g have a relative maximum? Justify your response.

e) For what value of x does g have an inflection point? Justify your response.

f) Graph $g(x)$