4.5 Worksheet
Solutions

1. Center the linearization at $x = 0$. (it's close to 0.1)
We need a point and a slope: Point $\left(0, \frac{1}{2}\right)$ and the slope is obtained by plugging $x = 0$ into the derivative which is given by

$$f\,'(x) = -\frac{1}{2}(4+x)^{-3/2} = \frac{-1}{2(4+x)^{3/2}}.$$

Since $f\,'(0) = -\frac{1}{16}$, we have the equation of the line to be $y - \frac{1}{2} = \frac{-1}{16}(x-0)$ or $y = \frac{-1}{16}x + \frac{1}{2}$.

Therefore, $f\,(0.1) \approx \frac{-1}{16}(0.1) + \frac{1}{2} = \frac{79}{160} = .49375$

2. If $y = \sin\left(x^2 - 3\right)$, then $dy = \cos\left(x^2 - 3\right) \cdot 2x\ dx$. Substituting the given values we get $dy = \dfrac{\sqrt{3}}{5} \approx .3464101615$

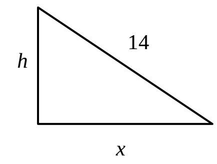
3. Given $r = 0.7$ and $dr = 0.01$. Asked to find dV. Volume of a sphere is given by $V = \dfrac{4}{3}\pi r^3$, so $dV = 4\pi r^2\ dr$.
Substituting the given values, we get $dV = 0.0196\pi \approx .061575216$

4. Asked to find percent error in measuring the side of a square, *ds*. With the constraint that the error in computing the area cannot exceed 2.5% … So, if $A = s^2$ then $dA = 2s\ ds$. Using the constraint $\left|\dfrac{dA}{A}\right| < .025$, we can substitute to get $\left|\dfrac{2s\ ds}{s^2}\right| < .025$. Solving for $|ds|$, we obtain $|ds| < .0125s$.

MEANING … that the error in measuring the side of the square (*ds*) must be less than 1.25% of the ACTUAL side measurement.

Related Rates WS
SOLUTIONS

1. We want to find $\frac{dx}{dt}$ when $h = 6$ and we know that $\frac{dh}{dt} = -2$.

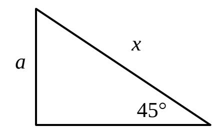


EQUATION: $x^2 + h^2 = 14^2$

$$2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$$

When $h = 6$, use the equation to find $x = \sqrt{160}$. Therefore, $\frac{dx}{dt} = \frac{12}{\sqrt{160}} \approx .9486832981$ ft/s.

2. We want to find $\frac{da}{dt}$ and we know that $\frac{dx}{dt} = 400$.



Notice that the angle is not changing, so we can find the sine of 45 degrees.

EQUATION:
$$\sin 45° = \frac{a}{x}$$
$$x \sin 45° = a$$
$$\frac{dx}{dt} \cdot \frac{\sqrt{2}}{2} = \frac{da}{dt}$$
$$400 \cdot \frac{\sqrt{2}}{2} = \frac{da}{dt}$$
$$282.8427125 \text{ mi/hr} \approx \frac{da}{dt}$$

3. We want to find $\frac{dV}{dt}$ when $r = 5$, and we know that $\frac{dr}{dt} = -10$.

EQUATION: $V = \frac{4}{3}\pi r^3$

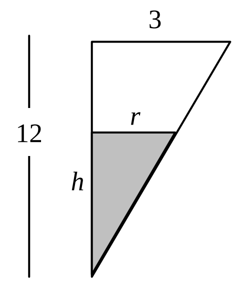
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$
$$\frac{dV}{dt} = 4\pi(5)^2(-10)$$
$$\frac{dV}{dt} = -1000\pi \text{ cm}^3/\text{min} \approx -3141.592654 \text{ cm}^3/\text{min}$$

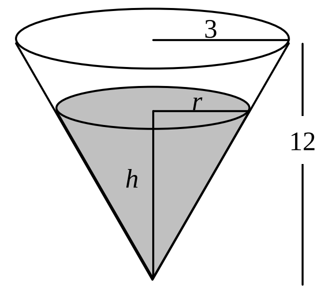
4. We want to find $\frac{dA}{dt}$ at the end of 8 seconds, and we know that $\frac{dr}{dt} = 4$. Since $\frac{dr}{dt} = 4$, at the end of 8 seconds, $r = 32$.

EQUATION: $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$
$$\frac{dA}{dt} = 2\pi(32)(4)$$
$$\frac{dA}{dt} = 256\pi \approx 804.2477193 \text{ ft}^2/\text{s}$$

5. We want to find $\frac{dh}{dt}$ when h = 6, and we know that $\frac{dV}{dt} = -3$.





EQUATION: $V = \frac{1}{3}\pi r^2 h$

But we need to write r in terms of h … using the similar triangles shown, we have $r = \frac{h}{4}$.

NEW EQUATION: $V = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h$

$$V = \frac{\pi}{48}h^3$$

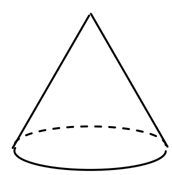
$$\frac{dV}{dt} = \frac{\pi}{16}h^2\frac{dh}{dt}$$

$$-3 = \frac{\pi}{16}(6)^2\frac{dh}{dt}$$

$$\frac{-4}{3\pi}\ {}^{\text{in}}\!/_{\text{min}} = \frac{dh}{dt}$$

$$-.4244131816\ {}^{\text{in}}\!/_{\text{min}} \approx \frac{dh}{dt}$$

6. We want to find $\frac{dV}{dt}$ when h = 15, and we know that $\frac{dh}{dt} = 4$ and $h = .5D = r$



EQUATION. $V = \frac{1}{3}\pi r^2 h$

$$V = \frac{1}{3}\pi(h)^2 h$$

$$V = \frac{1}{3}\pi h^3$$

$$\frac{dV}{dt} = \pi h^2\frac{dh}{dt}$$

$$\frac{dV}{dt} = \pi\left(15^2\right)(4)$$

$$\frac{dV}{dt} = 900\pi \approx 2827.433388\ {}^{\text{ft}^3}\!/_{\text{min}}$$

7. First, since the diameter and height of the original cone are both 10, you can determine the radius of the water to be half the height, which means $r = \frac{h}{2}$. (see problem #5)

a) Since the volume of the water is $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$, then when h = 5, $V = \frac{125\pi}{12} \approx 32.72492347$

b) We want to find $\frac{dV}{dt}$ when $h = 5$, and we know that $\frac{dh}{dt} = \frac{-3}{10}$. Since $V = \frac{\pi}{12}h^3$

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{4}\left(5^2\right)\left(\frac{-3}{10}\right)$$

$$\frac{dV}{dt} = \frac{-15\pi}{8} \approx -5.890486225 \ {}^{\text{cm}^3}\!/_{\text{hr}}$$

c) Rate of change in the volume being Directly proportional to the exposed SA of the water means that we need to show that $\frac{dV}{dt} = k \cdot (SA)$ for some value of k. Since the SA at any moment is just the area of a circle, we know

$$SA = \pi r^2 = \pi\left(\frac{h}{2}\right)^2 = \frac{\pi}{4}h^2.$$

We already found that $\frac{dV}{dt} = \frac{\pi}{4}h^2 \cdot \frac{dh}{dt}$. Substituting the above value into this equation we have $\frac{dV}{dt} = (SA)\cdot\frac{dh}{dt}$. Thus, $\frac{dh}{dt}$ is our value of k, and in this problem, we know that $k = \frac{-3}{10}$.