4.5 LINEARIZATION AND NEWTON'S METHOD

Linearization

The goal of linearization is to approximate a curve with a line. Why? Because it's easier to use a line than a curve! The basic idea of linearization is to find the equation of the tangent line at a certain point, and use the tangent line to estimate the desired value of the original function.

Example: Consider $f(x)=\sqrt{x}$. We all know that $f(4) = 2$, but without a calculator, what is $f(4.1)$?
Complete parts *a* and *b* to find a good approximation.

a) Find the equation of the tangent line for $f(x)$ when $x = 4$. Your book refers to this as $L(x)$.

b) The line you found is approximately the same as $f(x)$ "centered at $x = 4$". Use this to approximate $f(4.1)$.

c) Use a calculator to approximate $f(4.1)$. Are you close?

Example: [1969 Multiple Choice AB #36] The approximate value of $y=\sqrt{4+\sin x}$ at $x = 0.12$, obtained from the tangent to the graph at $x = 0$, is

A) 2.00 B) 2.03 C) 2.06 D) 2.12 E) 2.24

Example: [1973 Multiple Choice AB #44] For small values of *h*, the function $\sqrt[4]{16+h}$ is best approximated by which of the following?

A) $4+\frac{h}{32}$ B) $2+\frac{h}{32}$ C) $\frac{h}{32}$ D) $4-\frac{h}{32}$ E) $2-\frac{h}{32}$

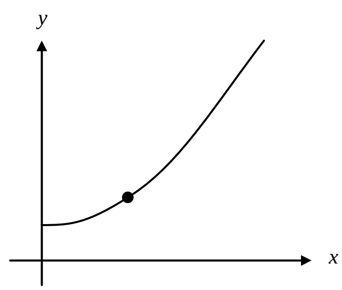
Example: [1998 Multiple Choice BC #92 … with calculator] Let *f* be the function given by $f(x) = x^2 - 2x + 3$. The tangent line to the graph of *f* at *x* = 2 is used to approximate the values of *f* (*x*). Which of the following is the greatest value for which the error resulting from this tangent line approximation is less than 0.5?

A) 2.4 B) 2.5 C) 2.6 D) 2.7 E) 2.8

Differentials

Approximations aren't exact! (Aren't you glad you woke up this morning to hear that enlightening bit of information?!) If we use a line to approximate a curve, it gives us a good estimate, as long as we don't go too far away from the center point. Wouldn't it be nice if we knew how far off our approximation is going to be? Well, whether you are excited about this or not, here we go!

Example: Consider the function *f* below. Label the point $(c, f(c))$, and draw the tangent line at that point.



Example: What is the equation of the *tangent line* you drew? Keep in mind … this is just a linearization of the curve.

Move a "small" distance to the right of *c*. Normally, we would call this distance Δx, but when Δx is very small, we will instead use the notation *dx*, the **differential of *x***.

Example: What is the **function** value at this point (when *x* = *c* + *dx*) ?

Example: What is the value of this point on the *tangent line* (when *x* = *c* + *dx*) ?

Example: How much did the *y*-values ACTUALLY change?
You are finding the change in *y* values for the ___________________.

Example: How much did the *y*-values APPROXIMATELY change?
You are finding the change in *y* values for the ___________________. (This is called *dy* … the differential in *y*)

In other words, if we were to use any value of *x*, the approximate change in *y* after a small change in *x* would be written

$$dy = f'(x)\,dx$$

This should look VERY familiar …

What differentials allow us to do is to say that *if the ratio of the differentials exists*, it will be equal to the derivative. It allows us to write $\frac{dy}{dx}$ as the derivative of *y* with respect to *x*, but use the *dy* and *dx* as <u>*separate terms*</u>.
Finding a differential is very similar to finding a derivative.

Example: Find the differential *dy* if $y = x^3 - 5x$

Example: Find $d\left[\cos(5x)\right]$.

Since *dy* is the approximate change in the *y* values when *x* is changed a small amount, we can use differentials to estimate the change in other problems if we know the small change in *x*.

Example: Find the differential *dy* when *dx* = 0.01 and *x* = 2, if $y = x^5 - 4x^3$. Explain what you've found.

Example: Find the differential *dy* when *dx* = –0.2 and *x* = 1, if $y = x^2 e^x$. Explain what you've found.

Example: Without a calculator, use differentials to approximate $\sqrt{4.2}$.

As we move from a point *c* to a nearby point *c* + *dx*, we can describe the change in *f* three ways:
Please don't memorize this … understand the vocabulary!

	ACTUAL	**ESTIMATED**
Absolute change		
Relative change		
Percentage change (just relative change made into a %)		

Example: The range R of a projectile is $R = \frac{v_0^{\ 2}}{32}(\sin 2\theta)$, where $v_0^{\ 2}$ is the initial velocity in feet per second and θ is the angle of elevation. If $v_0 = 2200$ feet per second and θ changed from 10° to 11°, use differentials to approximate the change in the range.

Example: The measurement of a side of a square is found to be 15 centimeters. The possible error in measuring the side is 0.05 centimeter.

a) Approximate the percent error in computing the area of the square.

b) Estimate the maximum allowable percent error in measuring the side if the error in computing the area cannot exceed 2.5%.

c) Find the error.

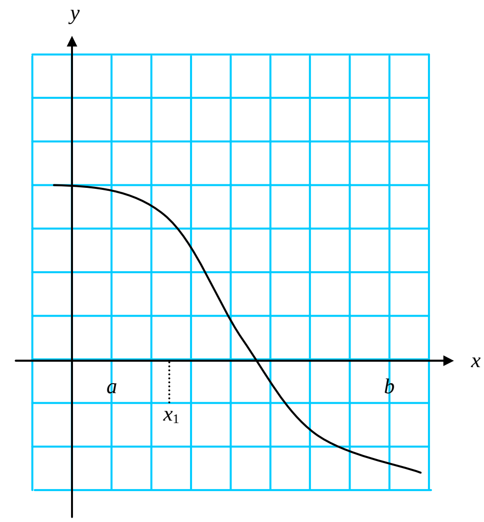
Example: A surveyor standing 50 feet from the base of a large tree measures the angle of elevation to the top of the tree as 71.5°. How accurately must the angle be measured if the percent error in estimating the height of the tree is to be less than 6%?

Notecards from Section 4.5: Linearization; Differentials; Absolute vs Relative vs Percentage Change

Newton's Method
… (no longer part of the AB curriculum) … time permitting …

Newton's method is a process using linearization to approximate (amazingly accurately) the zeroes of a function.

Example: Consider a continuous, differentiable function such that $f(a) > 0$ and $f(b) < 0$. What does the Intermediate Value theorem guarantee must occur between *a* and *b*?



Newton's method is based on the assumption that the tangent line at a point crosses the *x* axis at *about* the same place as the function. Since it is relatively easy to calculate the *x* – intercept of a line, we use this line to create a new estimate of the zero.

Example: Draw a tangent line to the function value when $x = x_1$.

Example: **Label** the *x* – intercept of the tangent line x_2. **Locate** the function value at x_2. **Draw** the tangent line at this point and **find** the *x* – intercept of the line.

Example: Is this point close to the point where the original function crosses the *x* – axis?

Example: Repeat the process one more time. How close do you get to the actual *x* – intercept?

Example: Can you think of a situation where this method would NOT work?

Let's look at this process algebraically. Beware, it's going to get a little messy, but the end result ain't that bad!
First thing we did was locate a point $(x_1, f(x_1))$ and draw the tangent line.

Example: In point – slope form, what is the equation of the tangent line at the point $(x_1, f(x_1))$?

Example: Solve this equation for *y*.

Second thing we did was find the *x* – intercept of this tangent line.
Example: Find the *x* – intercept of this line. Remember, x_1 is a constant, not a variable! (Let $y = 0$ and solve for *x*).

Once we had the *x* – intercept, we repeated the process, until we get as close as we want. So we will call the *x* – intercept we obtained in the last example x_2.

Example: Since $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$, we can use this to obtain an estimate for x_3. What would that estimate be?

Example: Remind yourself what these values are again!

Repeated application of this process is called NEWTON'S METHOD! Each successive application of this procedure is called an **iteration**.
Let's try one! ☺

Example: Use Newton's method to approximate the zeros of $f(x) = 2x^3 + x^2 - x + 1$. Continue the iterations until two successive approximations differ by less than 0.0001.