4.2 MEAN VALUE THEOREM

Notecards from Section 4.2: Mean Value Theorem; Relationship of f' to Increasing/Decreasing

The Mean Value Theorem is considered by some to be the most important theorem in all of calculus. It is used to prove many of the theorems in calculus that we use in this course as well as further studies into calculus.

The Mean Value Theorem

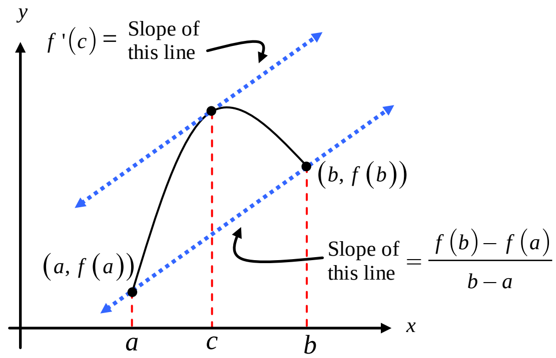
If f is continuous on the closed interval [a, b] and differentiable on the open interval (a, b), then *<u>there exists</u>* a number c in (a, b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Just like the Intermediate Value Theorem, this is an *existence theorem*. The Mean Value Theorem does not tell you what the value of c is, nor does it tell you have many exist. Again, just like the Intermediate Value Theorem, you must keep in mind that c is an x – value.

Also, the hypothesis of the Mean Value Theorem (MVT) is highly important. If any part of the hypothesis does not hold, the theorem cannot be applied.

Basically, the Mean Value Theorem says, that the average rate of change over the entire interval is equal to the instantaneous rate of change at some point in the interval.



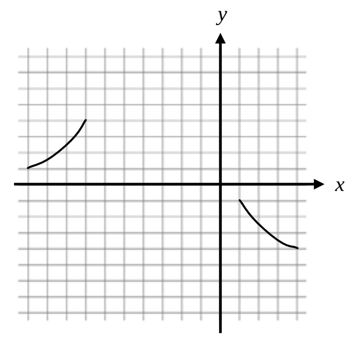
Example 1: A plane begins its takeoff at 2:00 pm on a 2500-mile flight. The plane arrives at its destination at 7:30 pm (ignore time zone changes). Explain why there were at least two times during the flight when the speed of the plane was 400 miles per hour.

Example 2: Apply the Mean Value Theorem to the function on the indicated interval. In each case, make sure the hypothesis is true, then find all values of c in the interval that are guaranteed by the MVT.

a) $f(x)=x(x^2-x-2)$ on the interval [–1, 1]

b) $f(x)=\dfrac{x+1}{x}$ on the interval [0.5, 2].

Example 3: The figure to the right gives two parts of the graph of a differentiable Function f on [–10, 4]. The derivative $f\,'$ is also continuous.



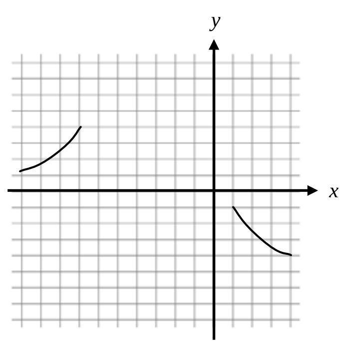
a) Explain why f must have at least one zero in (–7, 1).

b) Explain why there must be at least one point in the interval (–7, 1) whose derivative is $-\frac{5}{8}$.

c) Explain why $f\,'$ must also have at least one zero in the interval [–10, 4]. What are these zeros called?

d) Make a possible sketch of the function with one zero of $f\,'$ on the interval [–10, 4].

e) Make a possible sketch below of the function with exactly <u>two</u> zeros of $f\,'$ on the interval [–10, 4].



While the Mean Value Theorem is used to prove a wide variety of theorems, we will be focusing on the results and/or consequences of the Mean Value Theorem. In this section, we will discuss when a function increases and decreases as well as a brief introduction to antiderivatives.

Increasing versus Decreasing

Why mathematicians feel the need to define everything is a mystery you will probably never figure out unless you become one. Then, for some inexplicable reason, you will find yourself questioning the truthfulness of every argument ever made, reducing every argument to its basic foundational vocabulary, and finally analyzing the very soul and fiber of the definitions. With this in mind, we will now define what it means for a function to be increasing and decreasing. (Obviously, we cannot just say that a function is increasing when all the function values get bigger.)

Definitions of Increasing and Decreasing Functions

A function *f* is **increasing** on an interval if for any two numbers x_1 and x_2 in the interval,

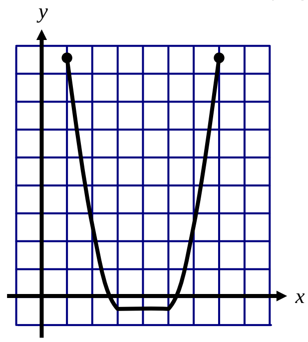
$$x_1 < x_2 \text{ implies } f(x_1) < f(x_2).$$

A function *f* is **decreasing** on an interval if for any two numbers x_1 and x_2 in the interval,

$$x_1 < x_2 \text{ implies } f(x_1) > f(x_2).$$

Example 4: What interval is the function decreasing? increasing? constant?

Example 5: What is the value of the derivative when the function is decreasing? increasing? constant?



Test for Increasing and Decreasing Functions

Let *f* be a function that is continuous on the closed interval [*a*, *b*] and differentiable on the open interval (*a*, *b*).

1. If $f'(x) > 0$ <u>for all</u> *x* in (*a*, *b*), then *f* is increasing on [*a*, *b*].
2. If $f'(x) < 0$ <u>for all</u> *x* in (*a*, *b*), then *f* is decreasing on [*a*, *b*].
3. If $f'(x) = 0$ <u>for all</u> *x* in (*a*, *b*), then *f* is constant on [*a*, *b*].

Guidelines for Finding Intervals on Which a Function is Increasing or Decreasing

Let *f* be continuous on the interval (*a*, *b*). To find the open intervals on which *f* is increasing or decreasing use the following steps:

1. Find the <u>critical points</u> of *f* in the interval (*a*, *b*), and use these numbers to create a sign chart.
2. Determine the sign of $f'(x)$ at ONE test value in each interval. (label your sign chart)
3. Use the <u>signs of the derivative</u> to determine whether or not the function is increasing or decreasing.
4. Your sign chart is not enough to justify your response. Your response should be worded …
   "The function is increasing (or decreasing) on the interval (*c*, *d*) since $f'(x) > 0$ (or $f'(x) < 0$) "

Example 6: Find the intervals on which $f(x) = x^3 - 12x - 5$ is increasing or decreasing.

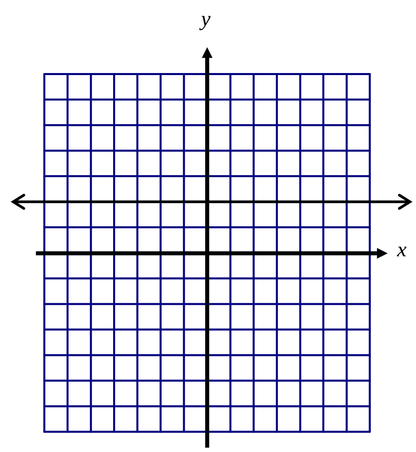
Example 7: Find the intervals on which $f(x)=\left(x^2-9\right)^{2/3}$ is increasing or decreasing.

Antiderivatives

Example 8: Suppose you were told that $f'(x)=2x-1$. What could $f(x)$ possibly be? Is there more than one answer?

Finding the function from the derivative is a process called ***antidifferentiation***, or finding the antiderivative.

Example 9: Suppose the graph of $f'(x)$ is given to the right. Draw at least three possible functions for $f(x)$. (*Hint*: If the derivative is given, then the *y*-values of the derivative are the slopes of the original function.)



The three functions you drew should only differ by a constant. If you let C represent this constant, then you can represent the *family* of all antiderivatives of $f'(x)$ to be $f(x)=2x+C$.

Example 10: If you were told that $f(3)=-2$, what would the value of C be?

IMPORTANT ♬: If a function has one antiderivative it has many antiderivatives that all differ by a constant. Unless you know something about the original function, you cannot determine the exact value of that constant, but it must be in your answer!