4.1 EXTREME VALUES OF FUNCTIONS

Notecards from Section 4.1: Where to Find Extrema, Optimization

Extrema (plural for extremum) are the maximum or minimum values of functions. We need to distinguish between absolute extrema and relative extrema, and how to locate them. You have used your calculator in the past to calculate a maximum or minimum value. In this course, however, you must use calculus reasons to find maximums and minimums!

Definition of Absolute Extrema … The BIGGEST or smallest y-value in the interval.

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum of *f* on the interval *I*** if $f(c) \le f(x)$ for all x in I.
2. $f(c)$ is the **maximum of *f* on the interval *I*** if $f(c) \ge f(x)$ for all x in I.

The Extreme Value Theorem

If f is continuous on a closed interval [a, b], then f has both a minimum and a maximum on the interval

Example 1: Draw a sketch, and find the absolute extrema of the function $f(x) = \sqrt{4-(x+2)^2}$ on the interval
If no maximum or minimum exists, which part of the extreme value theorem hypothesis isn’t satisfied?

(a) [–4, 0]

(b) [–2, 0)

(c) (–4, –2)

(d) [1, 2)

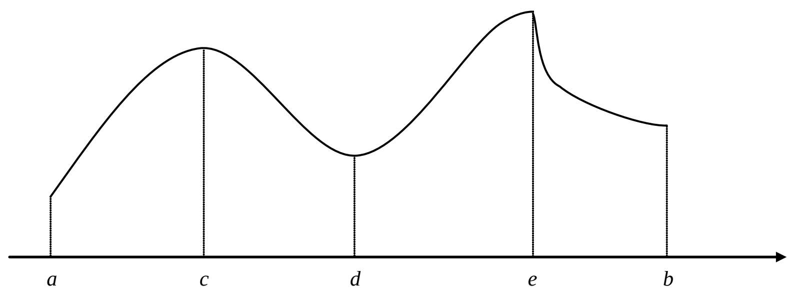
Relative Extrema and Critical Numbers

Definition of Relative Extrema

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum**.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum**.

Basically relative extrema exist when the value of the function is larger (or smaller) than all other function values relatively close to that value.

Example 2: Suppose you were given the function below. Label $f(a)$ through $f(e)$ as absolute or relative extrema.



When given a graph it is fairly simple to identify the extrema. The question to be asked then is how do we find the extrema when we do not have a graph given to us?

Example 3: Except at the endpoints *a* and *b*, what do you notice about the derivative at the relative extrema in the last example?

Definition of a Critical Point

Let f be defined at *c*. If $f'(c) = 0$ or if f' is undefined at *c*, then *c* is a **critical point** of *f*.

Relative Extrema Occur Only at Critical Points

If f has a relative minimum or relative maximum at $x = c$, then *c* is a critical number of *f*.

IMPORTANT ♬: Just because the derivative is equal to zero (or undefined) does not mean there is a relative maximum or minimum there. *Relative extrema* can occur ONLY at critical points, and critical points occur ONLY when the derivative is either 0 or undefined, however, it is possible for the derivative to equal zero (or undefined) and there be NO extrema there.

***If you need to convince yourself of this, try $f(x) = x^3$ and $f(x) = \sqrt[3]{x}$ at $x = 0$. …
Is the derivative zero? Undefined? … Is there a maximum or a minimum?

Guidelines for Finding Absolute/Relative Extrema on a Closed Interval

1. Find the critical numbers of f in (a, b). Do this by ____.

2. Evaluate f at each critical number in (a, b).

3. Evaluate f at the endpoints of $[a, b]$. DON'T FORGET THIS PART IF IT THE INTERVAL IS CLOSED!!!!

4. The least of the values from steps 2 and 3 is the absolute minimum, and the greatest of these values is the absolute maximum. If the interval is closed and the endpoints do no result in an absolute max or min, a sign chart can be used to determine whether or not the endpoints result in a relative max or min.

Example 4: The critical numbers (or critical points) are _______ values, while the maximums/minimums of the function are _______ values. In other words, if the point (2, 70) is a relative minimum, the minimum of the function is ________, and it occurs at ________ . (This is how you correctly describe extrema.)

Example 5: Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval [–1, 2]. Use your graphing calculator to investigate first.

Example 6: Find the extrema of $g(x) = 2x - 3x^{2/3}$ on the interval [–1, 3]. Use your graphing calculator to investigate first.

Optimization … really §4.4

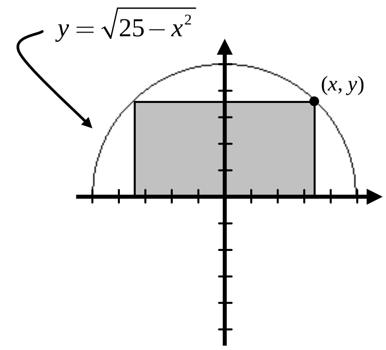
All optimization refers to is finding the absolute maximum or minimum. The usual difference between "optimization" problems and problems we have done so far is that the equation must be written by you.

Remember, ***if you have a closed interval you MUST test the endpoints of the function as well.***

Steps for Solving Applied Optimization Problems

Step 1: Understand the problem. Read it carefully, and ask yourself, "Self, what is the quantity to be maximized or minimized? What are the quantities which it depends on?"
Step 2: Draw a diagram if possible. Sometimes, more than one diagram helps you to determine how all the quantities are related. Identify all quantities from step 1 on your diagram.
Step 3: Assign variables to the quantities from step 1.
Step 4: Determine a function for the quantity to be optimized.
Step 5: Eliminate all but ONE variable and determine the domain of the resulting function. If this is necessary, you will need a second equation that relates the variables in the original equation.
Step 6: Optimize the function. (Find the absolute MAX or MIN)
(a) Calculate the derivative and find the critical numbers
(b) If the domain is a closed interval, compare the function's value of the critical numbers with that of the endpoints.
(c) If the domain is an open interval (or infinite interval), use the *first or second derivative tests* to analyze the behavior of the function. (*This will be explained in the next few sections*)

Example 7: A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$. What length and width should the rectangle have so that its area is a maximum?



Example 8: You are in a rowboat on Lake Perris, 2 miles from a straight shoreline taking your potential in-laws for a boat ride. Six miles down the shoreline from the nearest point on shore is an outhouse. You suddenly feel the need for its use. It is November, so the water is too cold to go in, and besides, your in-laws are already pretty unimpressed with your “yacht”. It wouldn’t help matters to jump over the side and relieve your distended bladder. Also, the shoreline is populated with lots of houses, all owned by people who already have restraining orders against you (apparently you've been out here before!). If you can row at 2 mph and run at 6 mph (you can run faster when you don’t have to keep your knees together), for what point along the shoreline should you aim in order to minimize the amount of time it will take you to get to the outhouse? (…And you thought calculus wasn’t useful!)