4.1 EXTREME VALUES OF FUNCTIONS

Extrema (plural for extremum) are the maximum or minimum values of functions. We need to distinguish between absolute extrema and relative extrema, and how to locate them. We have already looked at quadratic functions and you have used your calculator to find the extrema in the past.

Definition of Absolute Extrema

Let f be defined on an interval I containing c.

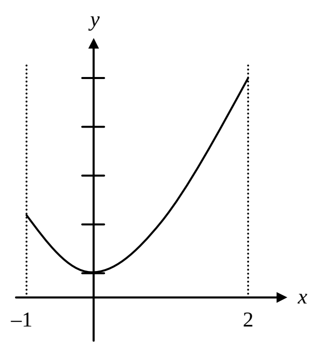
1. $f(c)$ is the **minimum of *f* on the interval *I*** if $f(c) \le f(x)$ <u>for all x</u> in I.
2. $f(c)$ is the **maximum of *f* on the interval *I*** if $f(c) \ge f(x)$ <u>for all x</u> in I.

The Extreme Value Theorem

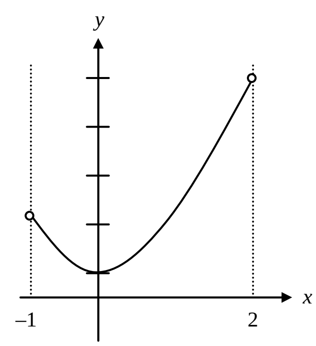
If f is continuous on a closed interval [a, b], then f has both a minimum and a maximum on the interval

Example: Using the graphs provided, find the minimum and maximum values on the given interval. If there is no maximum or minimum value, explain which part of the hypothesis of the Extreme Value Theorem is not satisfied.

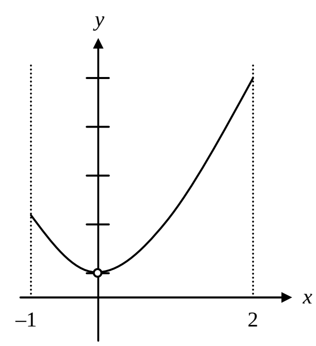
(a) [–1, 2]



(b) (–1, 2)



(c) [–1, 2]



Example: Draw a sketch to find the absolute extrema of the function $f(x) = \sqrt{4 - x^2}$ on the interval

(a) [–2, 2]

(b) [–2, 0)

(c) (–2, 2)

(d) [1, 2)

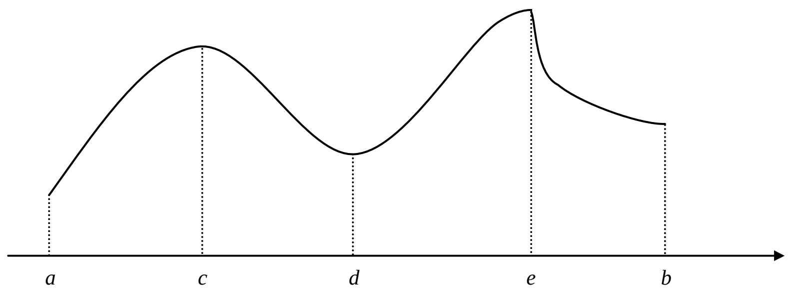
Relative Extrema and Critical Numbers

Definition of Relative Extrema

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum**.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum**.

Basically relative extrema exist when the value of the function is larger (or smaller) than all other function values relatively close to that value.

Example: The maximum and minimum points in the last example occurred at either the endpoints or at points interior to the interval. Suppose our function looked like the graph below. Label the points *a* – *e* as absolute or relative extrema.



When given a graph it is fairly simple to identify the extrema. The question to be asked then is how do we find the extrema when we do not have a graph given to us?

Example: Except at the endpoints *a* and *b*, what do you notice about the derivative at the relative extrema in the last example?

Definition of a Critical Point

Let f be defined at c. If $f'(c) = 0$ <u>or</u> if f' is undefined at c, then c is a **critical point** of f.

Relative Extrema Occur Only at Critical Points

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

IMPORTANT ♬: Just because the derivative is equal to zero (or undefined) does not mean there is a relative maximum or minimum there. *Relative extrema* can occur ONLY at critical points, and critical points occur ONLY when the derivative is either 0 or undefined, however, it is possible for the derivative to equal zero (or undefined) and there be NO extrema there.
***If you need to convince yourself of this, try $f(x) = x^3$ and $f(x) = x^{1/3}$ at $x = 0$. …
Is the derivative zero? Undefined? … Is there a maximum or a minimum?

Guidelines for Finding Absolute/Relative Extrema on a Closed Interval

1. Find the critical numbers of f in (a, b). Do this by ____.

2. Evaluate f at each critical number in (a, b).

3. Evaluate f at the endpoints of $[a, b]$. DON'T FORGET THIS PART IF IT THE INTERVAL IS CLOSED!!!!

4. The least of the values from steps 2 and 3 is the minimum, and the greatest of these values is the maximum. For now, a quick check of the values on either side of the critical number will help determine whether or not the function value at that critical number is a relative maximum, minimum, or neither.

Example: Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval [–1, 2]. Use your graphing calculator to investigate first.

Example: Find the extrema of $g(x) = 2x - 3x^{2/3}$ on the interval [–1, 3]. Use your graphing calculator to investigate first.

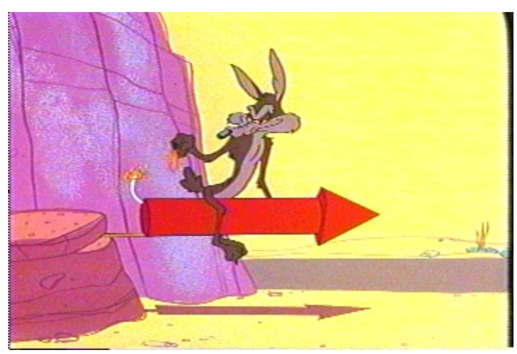
Example: Find the extrema of $h(\theta) = 2\sin\theta - \cos(2\theta)$ for $0 \le \theta \le 2\pi$. Use your graphing calculator to investigate first.

Example: The critical numbers (or critical points) are _______ values, while the maximums/minimums of the function are _______ values. In other words, if you are using the point (2, 70), the maximum/minimum of the function is ________, and it occurs at _______ . (This is how you correctly describe extrema.)

Example: Wile E. is after Road Runner again! This time he's got it figured out. Sitting on his ACME rocket he hides behind a hill anxiously awaiting the arrival that "beeping" bird. In his excitement to light the rocket he tips the rocket up. Instead of thrusting himself parallel to the ground where he can catch the Road Runner, he sends himself widely into the air following a path given by the position function

$$s(t) = -.00086x^4 + .067x^3 - 1.67x^2 + 14.77x.$$

How high does Wile E. go, and when does he reach that height?



Optimization … really §4.4

The concepts of finding the maximum or minimum of a function lay the foundation for the mathematical theory behind optimization. *To optimize something means to maximize or minimize some part of it.*
Remember, ***if you have a closed interval you MUST test the endpoints of the function as well.***

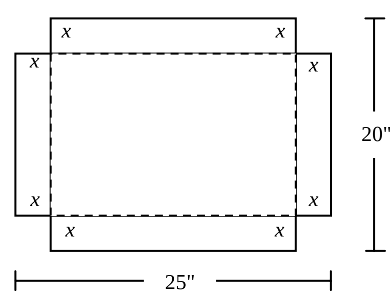
In every optimization problem, your goal is to find the maximum or minimum value of a function representing some real-world quantity. Your first (and often the hardest) task is always to find an expression for the function to be optimized. This involves translating the problem into mathematical terms. Once you have the function, you can then apply the methods we have learned to determine the maximum or minimum value.

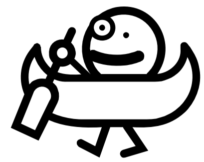
Steps for Solving Applied Optimization Problems

Step 1: Understand the problem. Read it carefully, and ask yourself, "Self, what is the quantity to be maximized or minimized? What are the quantities which it depends on?"
Step 2: Draw a diagram if possible. Sometimes, more than one diagram helps you to determine how all the quantities are related. Identify all quantities from step 1 on your diagram.
Step 3: Assign variables to the quantities from step 1.
Step 4: Determine a function for the quantity to be optimized.
Step 5: Eliminate all but ONE variable and determine the domain of the resulting function. This usually requires a second equation.
Step 6: Optimize the function.
(a) Calculate the derivative and find the critical numbers
(b) If the domain is a closed interval, compare the function's value of the critical numbers with that of the endpoints.
(c) If the domain is an open interval (or infinite interval), use the *first or second derivative tests* to analyze the behavior of the function. (*This will be explained in the next few sections*)

Example: Find two nonnegative real numbers that add up to 66 and such that their product is as large as possible.

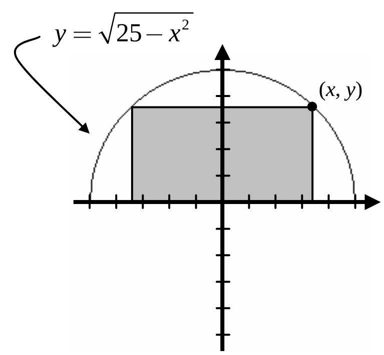
Example: An open-top box is to be made by cutting congruent squares of side length x from the corners of a 20- by 25-inch sheet of tin and bending up the sides (see figure below). How large should the squares be to make the box hold as much as possible? What is the resulting volume?





Example: You are in a rowboat on Lake Perris, 2 miles from a straight shoreline taking your potential in-laws for a boat ride. Six miles down the shoreline from the nearest point on shore is an outhouse. You suddenly feel the need for its use. It is November, so the water is too cold to go in, and besides, your in-laws are already pretty unimpressed with your "yacht". It wouldn't help matters to jump over the side and relieve your distended bladder. Also, the shoreline is populated with lots of houses, all owned by people who already have restraining orders against you (apparently you've been out here before!). If you can row at 2 mph and run at 6 mph (you can run faster when you don't have to keep your knees together), for what point along the shoreline should you aim in order to minimize the amount of time it will take you to get to the outhouse? (…And you thought calculus wasn't useful!)

Example: A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$. What length and width should the rectangle have so that its area is a maximum?



Notecards from Section 4.1: Where to Find Extrema, Optimization