3.9 Derivatives of Exponential and Logarithmic Functions

### Derivative of \( f(x) = e^x \)

\[
\frac{d}{dx}[e^x] = e^x
\]

**Proof**: Prove this derivative using the limit definition of the derivative and the fact that \( \lim_{h \to 0} \frac{e^h - 1}{h} = 1 \).

### The Chain Rule and \( f(x) = e^x \)

If \( u \) is a differentiable function of \( x \), then

\[
\frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}.
\]

**Example**: Find \( \frac{d}{dx}[e^{2x-1}] \)

**Example**: Find \( \frac{d}{dx}[e^{-x}] \)

**Example**: Suppose \( 10 = e^w + x^2 + y^2 \), find \( \frac{dy}{dx} \).
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**Example:** Find \( g'(t) \) if \( g(t) = t^e(\mathrm{e}^t) \)

**Derivative of** \( f(x) = \ln x \)

\[
\frac{d}{dx}[\ln x] = \frac{1}{x}
\]

**Proof:**

**The Chain Rule and** \( f(x) = \ln x \)

If \( u \) is a differentiable function of \( x \), then

\[
\frac{d}{dx}[\ln u] = \frac{1}{u} \frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[\ln u] = \frac{u'}{u}
\]

**Example:** Let \( y = \ln(2x+2) \). Find \( y' \).

**Example:** Let \( f(x) = \ln(\tan x) \). Find \( f'(x) \)

**Example:** Find \( g'(t) \) if \( g(t) = \ln(\ln t) \).
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Example: If \( y = \tan u \), \( u = v - \frac{1}{v} \), and \( v = \ln x \), what is the value of \( \frac{dy}{dx} \) at \( x = e \)?

A 0  B \( \frac{1}{v} \)  C 1  D \( \frac{2}{v^2} \)  E \( \sec^2(e) \)

We can use the properties of logarithms to simplify some problems. Here’s a quick refresher on those properties.

| Definition of a logarithm: \( \log_b a = e \leftrightarrow b^a = e \) |
| 3 Rules of Logarithms: |
| 1. \( \log_b (MN) = \log_b (M) + \log_b (N) \) |
| 2. \( \log_b \left( \frac{M}{N} \right) = \log_b (M) - \log_b (N) \) |
| 3. \( \log_b (M^k) = k \cdot \log_b (M) \) |

Change of Base Formula: \( \log_b a = \frac{\log a}{\log b} \) or \( \frac{\ln a}{\ln b} \)

Example: Use the properties of logarithms to rewrite the function, then find the derivative of \( y = \log_5 \sqrt{x} \).

Example: Find \( h'(x) \) if \( h(x) = \ln \left( \frac{1 + e^x}{1 - e^x} \right) \).
**Logarithmic Differentiation**

By utilizing the rules of logarithms and implicit differentiation, you can turn an exponential equation into an equation involving logarithms that is usually easier to deal with.

**Example:** Find \( \frac{dy}{dx} \) if \( y = 2^x \).

**Example:** Find \( \frac{dy}{dx} \) if \( y = 3^x \).

**Example:** Make a conjecture on \( \frac{d}{dx} \left[ a^x \right] \), where \( a \) is a constant greater than 0 and not equal to 1.

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**Derivative of \( a^x \)**

If \( a > 0 \) and \( a \neq 1 \) and \( u \) is a differentiable function of \( x \), then

\[
\frac{d}{dx} \left[ a^x \right] = \ln a \cdot a^x \frac{du}{dx}
\]

**Example:** Use the technique of logarithmic differentiation to find \( \frac{dy}{dx} \) for \( y = \frac{x\sqrt{x^2 + 1}}{(x+1)^3} \).

**Example:** Find the first derivative for \( y = x^{\ln x} \)
Example: Find $y'$ if $y = \frac{x^3}{3^x}$ first using the quotient rule, then using logarithmic differentiation.

Challenge: Solve the following without using a calculator at all:
If $f(x) = (x^2 + 1)^{(2-3x)}$, then $f'(1) =$

A $-\frac{1}{2}\ln(8e)$  B $-\ln(8e)$  C $-\frac{3}{2}\ln(2)$  D $-\frac{1}{2}$  E $\frac{1}{8}$

Notecards from Section 3.9: Derivatives of Exponential Functions, Derivatives of Logarithmic Functions, Logarithmic Differentiation