3.9 DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Derivative of $f(x) = e^x$

$$\frac{d}{dx}\left[e^x\right] = e^x$$

Proof: Prove this derivative using the limit definition of the derivative and the fact that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

The Chain Rule and $f(x) = e^x$

If u is a differentiable function of x, then

$$\frac{d}{dx}\left[e^u\right] = e^u \cdot \frac{du}{dx}.$$

Example: Find $\frac{d}{dx}\left[e^{2x-1}\right]$

Example: Find $\frac{d}{dx}\left[e^{-3/x}\right]$

Example: Suppose $10 = e^{xy} + x^2 + y^2$, find $\frac{dy}{dx}$.

Example: Find $g'(t)$ if $g(t) = t^e(e^{-t})$

Derivative of $f(x) = \ln x$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

Proof:

The Chain Rule and $f(x) = \ln x$

If *u* is a differentiable function of *x*, then

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx} \qquad \ldots \text{ or } \ldots \qquad \frac{d}{dx}[\ln u] = \frac{u'}{u}$$

Example: Let $y = \ln(2x+2)$. Find y'.

Example: Let $f(x) = \ln(\tan x)$. Find $f'(x)$

Example: Find $g'(t)$ if $g(t) = \ln(\ln t)$.

Example: If $y = \tan u$, $u = v - \dfrac{1}{v}$, and $v = \ln x$, what is the value of $\dfrac{dy}{dx}$ at $x = e$?

A 0 B $\frac{1}{e}$ C 1 D $\frac{2}{e}$ E $\sec^2(e)$

We can use the properties of logarithms to simplify some problems. Here's a quick refresher on those properties.

Definition of a logarithm: $\log_b a = c \quad \Leftrightarrow \quad b^c = a$

3 Rules of Logarithms:

1. $\log_b(MN) = \log_b(M) + \log_b(N)$
2. $\log_b\left(\dfrac{M}{N}\right) = \log_b(M) - \log_b(N)$
3. $\log_b(M^k) = k \cdot \log_b(M)$

Change of Base Formula: $\log_b a = \dfrac{\log a}{\log b}$ or $\dfrac{\ln a}{\ln b}$

Example: Use the properties of logarithms to rewrite the function, then find the derivative of $y = \log_5 \sqrt{x}$.

Example: Find $h'(x)$ if $h(x) = \ln\left(\dfrac{1+e^x}{1-e^x}\right)$.

Logarithmic Differentiation

By utilizing the rules of logarithms and implicit differentiation, you can turn an exponential equation into an equation involving logarithms that is usually easier to deal with.

Example: Find $\frac{dy}{dx}$ if $y = 2^x$.

Example: Find $\frac{dy}{dx}$ if $y = 3^x$.

Example: Make a conjecture on $\frac{d}{dx}\left[a^x\right]$, where *a* is a constant greater than 0 and not equal to 1.

Derivative of a^u

If $a > 0$ and $a \neq 1$ and *u* is a differentiable function of *x*, then

$$\frac{d}{dx}\left[a^u\right] = \ln a \cdot a^u \frac{du}{dx}$$

Example: Use the technique of logarithmic differentiation to find $\frac{dy}{dx}$ for $y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$.

Example: Find the first derivative for $y = x^{\ln x}$

Example: Find y' if $y = \frac{x^3}{3^x}$ first using the quotient rule, then using logarithmic differentiation.

Challenge: Solve the following without using a calculator at all:

If $f(x) = (x^2+1)^{(2-3x)}$, then $f'(1) =$

A $-\frac{1}{2}\ln(8e)$ B $-\ln(8e)$ C $-\frac{3}{2}\ln(2)$ D $-\frac{1}{2}$ E $\frac{1}{8}$

Notecards from Section 3.9: Derivatives of Exponential Functions, Derivatives of Logarithmic Functions, Logarithmic Differentiation