3.7 IMPLICIT DIFFERENTIATION

Implicit and Explicit Functions

Equations that are solved for y are called explicit functions, whereas equations that are not solved for y are called implicit.

For instance, the equation $x + 2y - 3 = 0$, implies that y is a function of x, even though it is not written in the form $y = -\frac{1}{2}x + \frac{3}{2}$. Up to this point in this class we have been using functions of x expressed in the form $y = f(x)$ such as

$$y = \frac{x+1}{x+2} \qquad \text{or} \qquad y = \sin x .$$

If we have an equation that involves both x and y in which y has not been solved for x, then we say the equation defines y as an *implicit* function of x. In this case, we may (or may not) be able to solve for y in terms of x to obtain an explicit function (or possibly several functions).

First a few skills that you will need …

Example 1: Find the derivative of the following expressions:

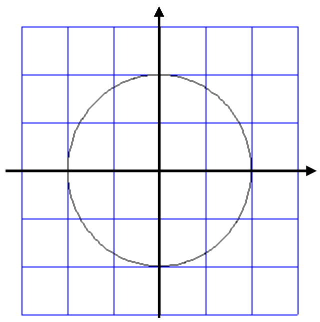
a) Find $\frac{dy}{dx}$: x^3

b) Find $\frac{dy}{dx}$: y^3

c) Find $\frac{dy}{dx}$: xy

d) Find $\frac{dy}{dx}$: $x^3 + y^2 - 3xy$

Example 2: Solve the equation $x^2 + y^2 = 4$ to obtain y written as an explicit function of x. The graph of this equation is a circle. What is the graph of each explicit function?



Example 3: Find $\frac{dy}{dx}$ for the circle $x^2 + y^2 = 4$:

a) by differentiating the explicit functions of x

b) by differentiating implicitly

c) Find where the derivative is positive, where it is negative, where it is zero, and where it is undefined.

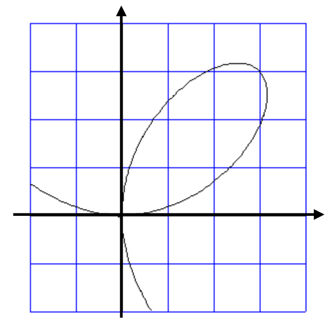
If we have y written as an explicit function of x, $y = f(x)$, then we know how to compute the derivative $\frac{dy}{dx}$. For an equation which defines y as an implicit function of x, we can compute the derivative $\frac{dy}{dx}$ without solving for y in terms of x with the following procedure. **The key to this entire procedure is to remember that even though you did not (or cannot) write *y* as a function of *x*, *y* is implicitly defined as a function of *x*.**

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation **with respect to *x*.** Remember, y is a function of x (use the Chain Rule)
2. Collect all $\frac{dy}{dx}$ terms on the left side of the equation and move all other terms to the right side of the equation.
3. Factor $\frac{dy}{dx}$ out of the left side of the equation.
4. Solve for $\frac{dy}{dx}$. (It is okay to have both x's and y's in your answer)

To find $\frac{dy}{dx}$ at a given point, plug both the x and y value into the equation you obtained in step 4.

Example 4: Given the curve $x^3 + y^3 = 6xy$ (shown to the right).



a) Find $\frac{dy}{dx}$.

b) Find the equation of the tangent line and normal (perpendicular) line to the graph at the point $\left(\frac{4}{3}, \frac{8}{3}\right)$.

Example 5: Find $\frac{dy}{dx}$ at (0, 0) of the function $\tan(x + y) = x$.