3.6 THE CHAIN RULE

Suppose you were asked to differentiate

$$h(x) = \sqrt{x^2 + 1}.$$

Up to this point in the course, we have no tools with which to differentiate this function. If we were to let

$$f(u) = \sqrt{u}, \text{ and}$$

$$u = g(x) = x^2 + 1,$$

then $h(x) = f(g(x)) = \sqrt{x^2+1}$. Differentiating each of the above functions separately, we obtain

$$\frac{df}{du} =$$

$$\frac{du}{dx} =$$

However, we need to have a rule that allows us to differentiate the original function, or more generally any composite function. Let us consider an example that may shed some light on how this might be accomplished.

Example: The length, *L*, in cm, of a steel bar depends on the air temperature, *H* °*C*, and the temperature *H* depends on time, *t*, measured in hours. If the length increases by 2 cm for every degree increase in temperature, and the temperature is increasing at 3 °*C* per hour, how fast is the length of the bar increasing? What are the units for your answer?

This last example illustrates a practical look at how the following theorem actually works.

The Chain Rule

If $y = f(u)$ is a differentiable function of *u* and $u = g(x)$ is a differentiable function of *x*, then $y = f(g(x))$ is a differentiable function of *x* and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or equivalently,

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\,g'(x).$$

Composite functions have an "inside function" and an "outside function". Another way to look at this would be

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\,g'(x)$$

Derivative of the "outside function" … leave the "inside function" alone.

Derivative of the "inside function"

The toughest part (at first) is learning to identify the "inside" and "outside" functions.

Example: Using *h* (*x*), from the top of the page, identify the "inside" and "outside" functions, then find $h'(x)$.

Example: Each of the following examples can be done without using the chain rule. First state how to find the derivative without using the chain rule, and then use the chain rule to differentiate.

(a) $f(x)=\dfrac{2}{3x+1}$

(b) $g(x)=(x+2)^3$

(c) $h(x)=\sin(2x)$

Using the Product, Quotient Rules, and Chain Rules

Example: Find $k'(x)$, if $k(x)=(x^2+1)\sqrt{2x-3}$.

Example: Find $\frac{dg}{dt}$, if $g=\left(\frac{t-2}{2t+1}\right)^9$

Example: Find $f'(x)$, if $f(x)=\tan(\cos x)$

Example: Find $\frac{dy}{dx}$, if $y=\sin\left(\tan\sqrt{\sin x}\right)$

Example: Find $\frac{ds}{d\theta}$, if $s=2\theta\sqrt{\sec\theta}$

Example: Find $\frac{dy}{dt}$, if $y = \tan x$. (No … there are no typos in this problem)

Example: For each of the following, use the fact that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$ to find $f'(5)$, if possible. If it is not possible, state what additional information is required.

a) $f(x) = g(x)h(x)$

b) $f(x) = g(h(x))$

c) $f(x) = \frac{g(x)}{h(x)}$

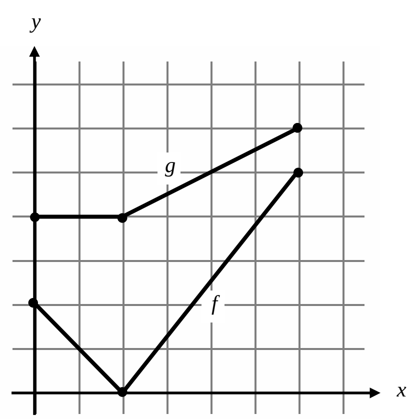
d) $f(x) = [g(x)]^3$ … ♬: Your book refers to $[g(x)]^3$ as $g^3(x)$

Example: A 15 – centimeter pendulum moves according to the equation

$$\theta = 0.2\cos(8t),$$

where θ is the angular displacement from the vertical in radians and t is the time in seconds. Determine the maximum angular displacement and the rate of change of θ when $t = 3$ seconds.

Example: Let $r(x) = f(g(x))$ and $s(x) = g(f(x))$ where f and g are shown in the figure at right.



a) Find $r'(1)$.

b) Find $s'(4)$.