3.4 VELOCITY AND OTHER RATES OF CHANGE

Instantaneous Rates of Change

We have already seen that the instantaneous rate of change is the same as the slope of the tangent line and thus the derivative at that point. Unless we use the phrase "average rate of change", we will assume that in calculus the phrase "rate of change" refers to the instantaneous rate of change.

Example: The length of a rectangle is given by $2t + 1$ and its height is $\sqrt{t}$, where t is time in seconds and the dimensions are in centimeters. Find the rate of change of the area with respect to time, and indicate the units of measure for this rate.

Example: Boyle's Law states that if the temperature of a gas remains constant, its pressure is inversely proportional to its volume. Show that the rate of change of the pressure is inversely proportional to the square of the volume.

Motion along a line

When a spring attached to a wall is stretched and then released, it moves back and forth. This motion can be described using functions involving sine and cosine (which we are not ready for just yet …). However, motion along a line in other circumstances (either horizontal or vertical lines) can also be described using functions. Typically, we use a position function $s(t)$ to describe the position s of an object after t seconds.

Important Vocabulary:

The *displacement* of an object is the TOTAL CHANGE IN POSITION.

The *average velocity* of the object is described as TOTAL CHANGE IN POSITION (displacement) divided by the TOTAL CHANGE IN TIME.

The *instantaneous velocity* of the object is the derivative of the position function. Unless we use the term "average velocity", we will assume that velocity refers to instantaneous velocity. A positive velocity indicates movement in the positive direction and a negative velocity indicates movement in the negative direction.

Speed is the absolute value of velocity. Thus speed is always positive value, whereas, velocity indicates direction.

Acceleration is the rate of change in velocity, implying then that acceleration is the derivative of velocity. Since it is the derivative of velocity, it is also the *second* derivative of position.

Example: Bugs Bunny has been captured by Yosemite Sam and forced to "walk the plank". Instead of waiting for Yosemite Sam to finish cutting the board from underneath him, Bugs finally decides just to jump. Bugs' position, s, is given by

$$s(t) = -16t^2 + 16t + 320$$

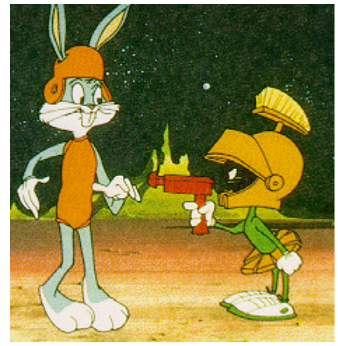
where s is measured in feet and t is measured in seconds.

a) When will Bugs hit the ground?

b) What is Bugs' velocity at impact? (What are the units of this value?)

c) What is Bugs' speed at impact?

d) Find Bugs' acceleration as a function of time. (What are the units of this value?)



Example: Once again trying to blow up earth because it interferes with his view of Venus, Marvin the Martian lands on the moon. Bugs Bunny, as always, interferes with his plan. Chasing Bugs, Marvin fires a warning shot straight up into the air with his Acme Disintegration Pistol. The height (in feet) after t seconds of the shot is given by

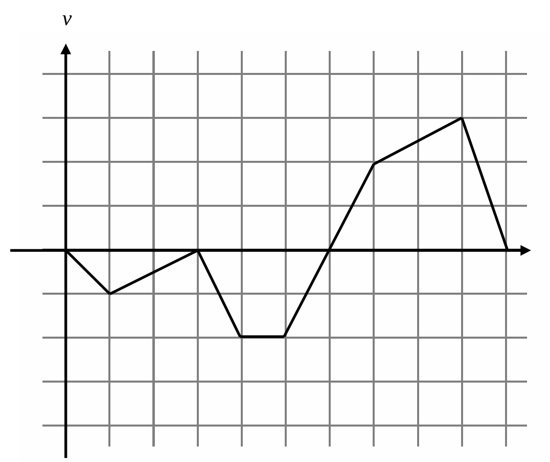
$$s(t) = -2.66t^2 + 135t + 3.$$

a) Find the velocity and acceleration as functions of time. (What is the meaning of the acceleration function?)

b) How long will it take for Marvin's shot to reach its maximum height?

c) What is the maximum height for Marvin's shot?

Example: Suppose the graph below shows the velocity of a particle moving along the x – axis.

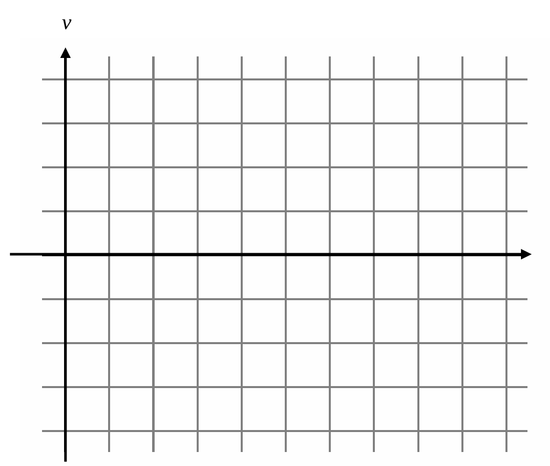


a) Which way does the particle move first?

b) When does the particle stop?

c) When does the particle change direction?

d) Graph the particle's acceleration for $0 < t < 10$.



d) When is the particle speeding up?

e) When is the particle slowing down?

f) When is the particle moving at a constant speed?

g) Graph the particle's speed for $0 \le t \le 10$.

Example: A particle moves along the x – axis so that its position at any time $t \geq 0$ is given by the function $x(t) = t^3 - 12t + 1$, where x is measured in feet and t is measured in seconds.

a) Find the displacement during the first 3 seconds.

b) Find the average velocity during the first 3 seconds.

c) Find the instantaneous velocity when $t = 3$ seconds.

d) Find the acceleration of the particle when $t = 3$ seconds.

e) When is the particle moving left?

f) At what value or values of t does the particle change directions?

Example: If the position of a particle on the x – axis at time t is $-5t^2$, then the average velocity of the particle for $0 \leq t \leq 3$ is

A) –45 B) –30 C) –15 D) –10 E) –5

Example: Rocket *A* has a positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \leq t \leq 80$ seconds as shown in the table below.

t (sec)	0	10	20	30	40	50	60	70	80
$v(t)$ (ft/sec)	5	14	22	29	35	40	44	47	49

a) Find the average acceleration of Rocket *A* over the time interval $0 \leq t \leq 80$ seconds. Indicate units of measure.

b) Using the data, find an estimate for $v'(35)$. Indicate units of measure.

Derivatives in Economics

Economists use calculus to determine the rate of change of costs with respect to certain factors. For instance, the *marginal cost of production* is the rate of change of cost with respect to the level of production. *Marginal cost* can also be viewed as the approximate cost of producing 1 more unit than is being produced now.

To see how this works, let us suppose that $c(x) = 4x + \frac{300,000}{x}$ is a function of the cost of producing x units of a product.

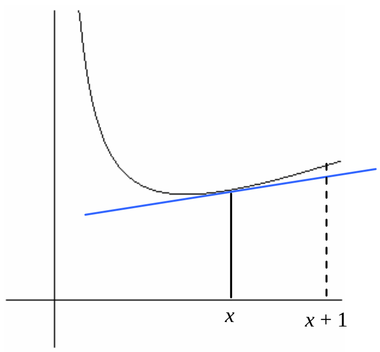
Example: If you were producing 100 units, what would the cost be?

Example: If you were to produce 101 units, what would the cost be?

Example: If you are currently producing 100 units, what is the actual cost of producing 1 more unit?

Example: If the function graphed below represents $c(x)$, then the slope of the tangent line at x is ____________.

The curve below represents the cost of production, $c(x)$. The line is the tangent line to the curve at x.



Example: Show what the ACTUAL change in cost would be to produce $x + 1$ units.

Example: Explain why $c'(x)$ approximates the change in cost to produce $x + 1$ units?

Example: What is the marginal cost to produce 101 units if you are currently producing 100 units?

Notecards from Section 3.4: Relationships between Position, Velocity, and Acceleration; Velocity vs. Speed