2.4 RATES OF CHANGE AND TANGENT LINES

Notecards from Section 2.4: Definition of Average Rate of Change, Definition of Instantaneous Rate of Change.

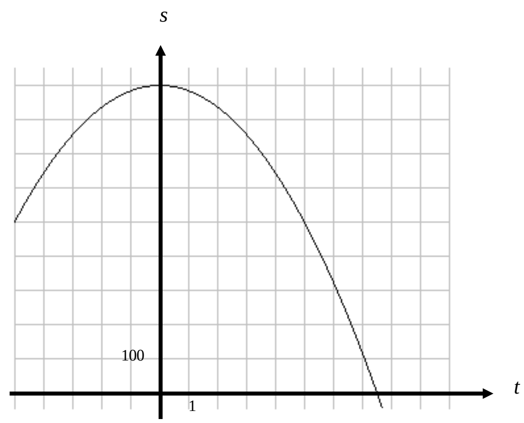
Average Rates of Change

Example 1: Remember this example? … Wile E. Coyote, once again trying to catch the Road Runner, waits for the nastily speedy bird atop a 900 foot cliff. With his Acme Rocket Pac strapped to his back, Wile E. is poised to leap from the cliff, fire up his rocket pack, and finally partake of a juicy road runner roast. Seconds later, the Road Runner zips by and Wile E. leaps from the cliff. Alas, as always, the rocket malfunctions and fails to fire, sending poor Wile E. plummeting to the road below disappearing into a cloud of dust.

Let's look at this problem from a graphical perspective. The equation that models Wile E.'s height at any time t is given by

$$s(t) = -16t^2 + 900$$

A graph of this equation is shown below.



a) Find the points on the graph that correspond to Wile E's position at $t = 0$ and $t = 5$ seconds.

b) Draw the line that passes through these points, and find the slope. What does this value mean?

The line you drew on the graph can be called a *secant line*. A secant line is a line through any two points on a curve. Just like we did in this example, we can always think of the **average rate of change** as the slope of the secant line. To find the slope of the secant line above we divided the total change in s by the total change in t. To find the average rate of change in the position (a.k.a. velocity) we found the total change in position divided by the total change in time.

We are able to find Wile E.'s average velocity for any period of time following the same procedure as above. Do you remember the problem we had finding the velocity of poor Wile E. Coyote at an exact moment in time? If we wanted to find the velocity of Wile E. Coyote at *exactly* 5 seconds, we tried to determine the average velocity using values of t that were closer and closer to $t = 5$.

c) Find Wile E.'s average velocity (rate of change) from $t = 4$ to $t = 5$ seconds. Graphically show this above.

d) Find Wile E.'s average velocity (rate of change) from $t = 4.5$ to $t = 5$ seconds. Graphically show this above.

e) Find Wile E.'s average velocity (rate of change) from $t = 4.9$ to $t = 5$ seconds. Graphically show this above.

d) What do you think would be the graphical interpretation of the velocity at exactly 5 seconds?

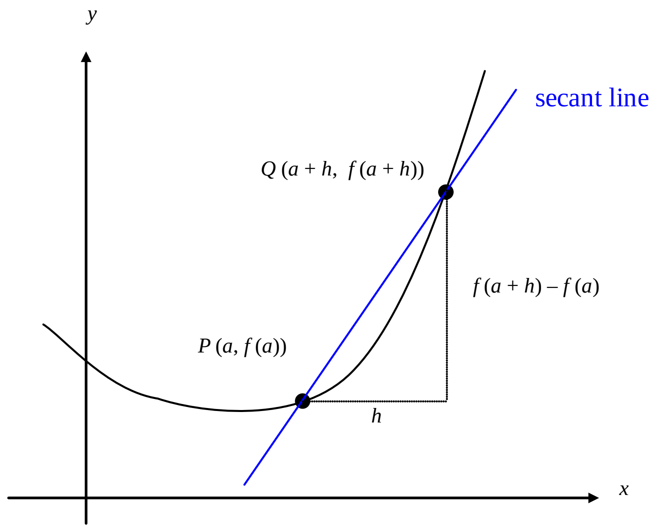
Slope of a Tangent Line … aka "Slope of a Curve at a Point"

For a general curve, the equation of the tangent line simply boils down to finding the *slope* of the tangent line. (Since we already have a point of tangency, if we knew the slope we would be able to write the equation of a line.)

We can *approximate* the slope of the tangent line using a **secant line**.

If $P\ (a, f(a))$ is the point of tangency we are concerned with, then we can pick an arbitrary point Q on the graph and estimate the tangent line at P using the slope of the secant line through P and Q.

Example 2: What is the slope of the secant line?



The beauty of this procedure is that you can obtain more and more accurate approximations to the slope of the tangent line by choosing points closer and closer to the point of tangency.

Example 3: How do we get closer and closer to the point of tangency?

Draw at least 3 more secant lines, using a point closer to P each time.

As $h \to 0$, the slope of the secant line approaches the slope of the tangent line.

Slope of a Curve

The slope of a line is always constant. The slope of a curve is constantly changing. Think of a curve as a roller coaster that you are riding. If for some reason the "track" were to just disappear, you would go flying off in the direction that you were traveling at that last instant before the track disappeared. The direction that you flew off to would be the slope of the curve at that point.

Using the slope of the secant line from the last example, we have the following definition.

Slope of a Curve at a Point

The **slope of the curve** $y = f(x)$ at the point $P\ (a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

♬: The expression $\dfrac{f(a+h) - f(a)}{h}$ is called a **difference quotient**.

Your goal is to SIMPLIFY the difference quotient, THEN evaluate the limit as h approaches 0.

The difference quotient is simplified when you have cancelled the h in the denominator in an algebraically correct way.

Finding the Equation of a Tangent Line

The **tangent line to the curve** at P is the equation of a line. Thus we need a point … P … and a slope … the limit as h goes to zero of the difference quotient.

The slope might not exist because the limit doesn't exist (discontinuity, vertical asymptote, oscillating function) or because the tangent line has a vertical slope.

** For a tangent line that exists, but has no slope, this definition doesn't quite fit. To cover the possibility of a vertical tangent line, we can use the following definition.

If f is continuous at a and

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\pm\infty$$

the vertical line, $x = a$ is a **vertical tangent line** to the graph of f.

Example 4: What types of graphs would have vertical tangent lines?

Example 5: Before we use this definition, be sure to become comfortable with the notation $f(a+h)$.

a) If $f(x)=\frac{1}{x}$, what is $f(a)$? … $f(a+h)$?

b) If $f(x)=x^2-4x$, what is $f(a)$? … $f(a+h)$?

c) If $f(x)=\sqrt{4x+1}$, what is $f(a)$? … $f(a+h)$?

Example 6: Back to example 1 … Let $f(x)=-16x^2+900$. Find the slope of the curve at $x = 5$.

♬ : This would be the **instantaneous velocity** of Wile E. Coyote at exactly 5 seconds!

Normal Line

The **normal line** to a curve at a point is the line perpendicular to the tangent at that point.

Example: Let $f(x) = \dfrac{1}{x+1}$.

a) Find the slope of the curve at $x = a$.

b) Find the slope of the curve at $x = 2$.

c) Write the equation of the tangent line to the curve at $x = 2$.

d) Write the equation of the normal line to the curve at $x = 2$.