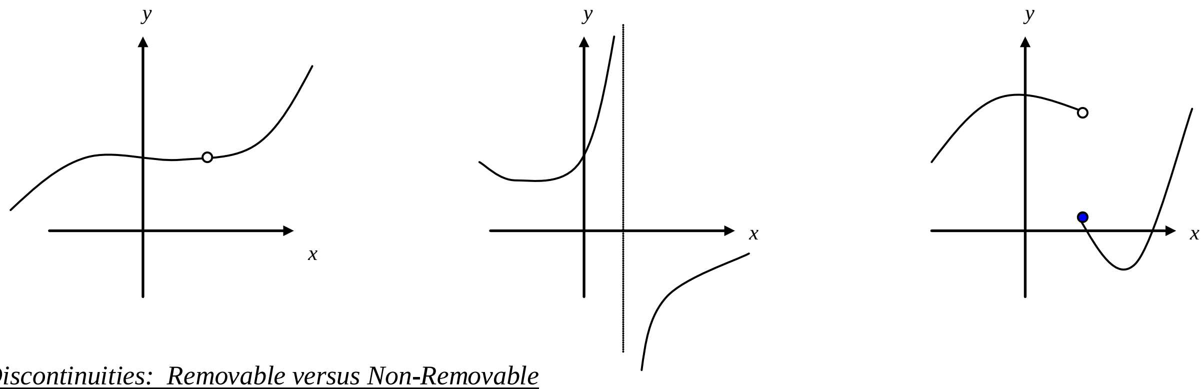
2.3 CONTINUITY

Notecards from Section 2.3: Definition of continuity at $x = c$, Types of Discontinuities, Intermediate Value Theorem

In §2.1 we referred to "well behaved" functions. "Well behaved" functions allowed us to find the limit by direct substitution. "Well behaved" functions turn out to be continuous functions. In this section we will discuss continuity at a point, continuity on an interval, and the different types of discontinuities.

In non – technical terms, a function is continuous if you can draw the function "without ever lifting your pencil". The following graphs demonstrate three types of discontinuous graphs.



Discontinuities: Removable versus Non-Removable

To say a function is discontinuous is not sufficient. We would like to know what type of discontinuity exists. If the function is not continuous, but I could make it continuous by appropriately defining or redefining $f(c)$, then we say that f has a **removable discontinuity**. Otherwise, we say f has a **non-removable discontinuity**.

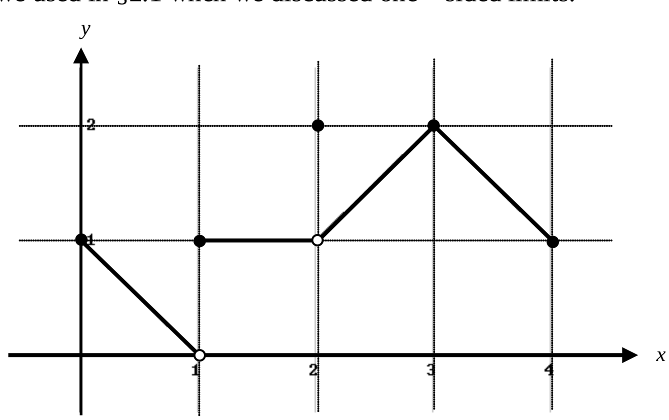
Once again, informally we say that f has a removable discontinuity if there is a "hole" in the function, but f has a non-removable discontinuity if there is a "jump" or a vertical asymptote.

**All polynomials are continuous.

For rational functions, we try to algebraically "remove" the discontinuity by canceling factors found in both the denominator and the numerator if possible. (Occurs when we get 0/0 we evaluating limits).

Example 1: Which (if any) of the three graphs above have a removable discontinuity?

Let's go back to the example we used in §2.1 when we discussed one – sided limits.



Example 2: Find the points (intervals) at which the function above is continuous, and the points at which it is discontinuous.

Definition: *Continuity at a Point*

Interior Point: A function $y = f(x)$ is **continuous at an interior point *c*** of its domain if

$$\lim_{x \to c} f(x) = f(c)$$

♬ : This last statement implies that $\lim_{x \to c} f(x)$ exists. This limit only exists if the limit from the left and right of c are equal! It also implies that the function value at c … $f(c)$ exists.

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint *a*** or is **continuous at a right endpoint *b*** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

Example 3: Go back to the last picture. For $c = 1$, 2, and 3, find $f(c)$, $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, and $\lim_{x \to c} f(x)$ if they exist. (Can you see how all the parts of the definition of continuity are important?)

Example 4: Discuss the continuity of each function

(a) $f(x) = \dfrac{1}{x-1}$

(b) $g(x) = \dfrac{2x^2 + x - 6}{x+2}$

(c) $h(x) = \begin{cases} -2x+3 & ; x < 1 \\ x^2 & ; x \geq 1 \end{cases}$

Using the Continuity Definition

Example 5: Determine the value of k such that the function is continuous on the entire real line.

$$g(x)=\begin{cases} x^2+7 & \text{if } x \geq 1 \\ x+k & \text{if } x<1 \end{cases}$$

Example 6: Determine the value of k in order to make the function continuous for all real numbers.

$$h(x)=\begin{cases} \dfrac{x^4-1}{x-1} & \text{if } x \neq 1 \\ k & \text{if } x=1 \end{cases}$$

Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Constant multiple: bf
2. Sum and difference: $f \pm g$
3. Product: fg
4. Quotient: $\dfrac{f}{g}$; $g(c) \neq 0$

The Intermediate Value Theorem (IVT)

If f is continuous on the closed interval [a, b] then f takes on every value between $f(a)$ and $f(b)$.

Suppose k is any number between $f(a)$ and $f(b)$, then there is at least one number c in [a, b] such that $f(c) = k$.

♬: The Intermediate value theorem tells you that at least one c exists, but it does not give you a method for finding c. This theorem is an example of an *existence theorem*.

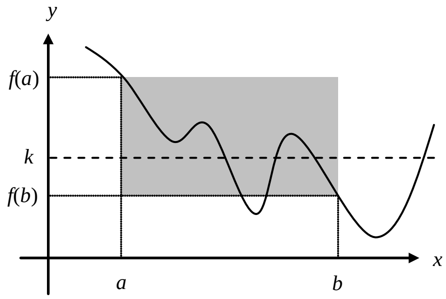
Example 7: In the Intermediate Value Theorem …

a) What are the necessary requirements in order to apply this theorem?

b) k is on which axis?

c) c is on which axis?

Example 8: Consider the function f below.



- Is f continuous on [a, b] ?
- Is k between $f(a)$ and $f(b)$?
- In this example, if $a < c < b$, then there are ________ c's such that $f(c) = k$.
- Label the c's on the graph as $c_1, c_2, \ldots$

Example 9: Let $f(x) = \dfrac{x^2 + x}{x - 1}$. Verify that the Intermediate Value Theorem applies to the interval $\left[\dfrac{5}{2}, 4\right]$ and find the value of c guaranteed by the theorem if $f(c) = 6$.