2.2 LIMITS INVOLVING INFINITY

Notecards from Section 2.2: Definition of a horizontal asymptote, Definition of a vertical asymptote, End Behavior, End Behavior Models, Oblique (Slant) Asympotes

We are going to look at two kinds of limits involving infinity. We are interested in determining what happens to a function as *x* approaches infinity (in both the positive and negative directions), and we are also interested in studying the behavior of a function that approaches infinity (in both the positive and negative directions) as *x* approaches a given value.

Finite Limits as $x \to \pm\infty$

Example 1: Investigate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = \frac{1}{x}$.

Example 2: Investigate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = \frac{2x-1}{x+3}$.

> *Definition: Horizontal Asymptote*
>
> The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either
>
> $$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

Rational Functions (two polynomial functions divided) have the same horizontal asymptote in both direction

Horizontal Asymptotes of Rational Functions

> *Definition: End Behavior Model*
>
> For a ***rational*** function $\frac{ax^m + \cdots}{bx^n + \cdots}$, where *m* is the degree of the numerator and *n* is the degree of the denominator. The end behavior model can be written as $\frac{ax^m}{bx^n}$ or $\frac{a}{b}x^{m-n}$.
>
> We can use the end behavior models of *rational* functions to identify any *horizontal asymptotes* the function.

If $f(x) = ax^m + \cdots$ and $g(x) = bx^n + \cdots$ then $\dfrac{f(x)}{g(x)}$ takes on three different forms.

	End Behavior Model	End Behavior	Asymptote
Degrees are equal $m = n$			
Larger degree in denominator $m < n$			
Larger degree in numerator $m > n$			

Example 3: For each example below do the following:

i) Write the end behavior model.
ii) Evaluate each limit.
iii) Determine whether or not there are any horizontal asymptotes. If so, what is the equation?
iv) Determine whether or not there are any slant (oblique) asymptotes. If so, what is the equation?

a) $\lim\limits_{x\to\infty} \dfrac{2x+5}{3x^2-6x+1}$

b) $\lim\limits_{x\to\infty} \dfrac{2x^2-3x+5}{x^2+1}$

c) $\lim\limits_{x\to\infty} \dfrac{x^4+x^3+9}{3x-3}$

<u>Functions that have more than one horizontal asymptote</u>

In the previous examples, if there was a limit as *x* approached positive infinity, the limit as *x* approached negative infinity was the same. This occurs whenever you have a rational function.

However, there are functions that have more than one horizontal asymptote.

Example 4: Investigate $\lim\limits_{x\to\infty} \dfrac{5+2^x}{3-2^x}$ and $\lim\limits_{x\to-\infty} \dfrac{5+2^x}{3-2^x}$

A general rule for functions that are divided (not necessarily rational functions) is that if the denominator "grows" faster than the numerator, the limit as x approaches infinity will be 0. If the numerator "grows" faster than the denominator, then as x approaches infinity, the limit will not exist.

Example 5: In the last section we proved that $\lim_{x\to 0}\frac{\sin x}{x}=1$. Investigate $\lim_{x\to\infty}\frac{\sin x}{x}$.

It should come as no surprise that the proof of this limit involves the Sandwich Theorem. ☺ See the text if you just can't contain your curiosity. The limit properties that we used when x approaches c are still valid as x approaches infinity.

Infinite Limits as $x \to a$

A second type of limit involving infinity is to determine the behavior of the function as x approaches a certain value when the function increases or decreases without bound.

Example: Investigate $\lim_{x\to 0^+}\frac{1}{x}$ and $\lim_{x\to 0^-}\frac{1}{x}$.

Definition: Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x)=\pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x)=\pm\infty$$

***This occurs whenever there is a value of x that gives you a 0 in the denominator (but not the numerator).

Important ♬: Infinity is NOT a number, and thus the limit FAILS to exist in both of these cases. If this seems confusing, then use the notation as $x \to a$ (from the right or left), then the function $f(x) \to \pm\infty$.

Example: Find the vertical asymptotes of $f(x)$. Describe the behavior of $f(x)$ to the left and right of each asymptote.

a) $f(x)=\frac{x^2-1}{2x+4}$

b) $f(x)=\frac{1-x}{2x^2-5x-3}$

c) $f(x)=\frac{x-2}{3x^2-5x-2}$