2.1 RATES OF CHANGE AND LIMITS

Notecards from Section 2.1: How to Evaluate a Limit, Properties of Limits, Definition of a Limit at $x = c$, When Limits fail, and Limits you should know.

Limits

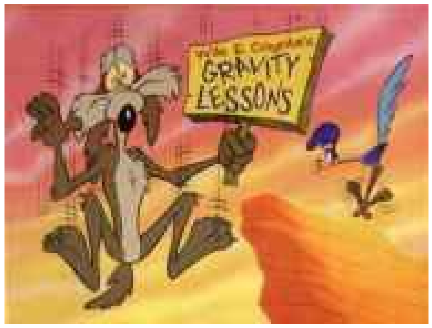
Limits are what separate Calculus from pre – calculus. Using a limit is also the foundational principle behind the two most important concepts in calculus, derivatives and integrals. Limits can be found using substitution, graphical investigation, numerical approximation, algebra, or some combination of these.

Average and Instantaneous Velocity

In pre – calculus courses, you used the formula $d = rt$ to determine the speed of an object. What you found was the object's average speed. A moving body's **average speed** during an interval of time is found by dividing the total distance covered by the elapsed time. (Speed is always positive … Velocity indicates direction and can be negative.) We are going to find the **average velocity**.

If an object is dropped from an initial height of h_0, we can use the position function $s(t) = -16t^2 + h_0$ to model the height, s, (in feet) of an object that has fallen for t seconds.

Example: Wile E. Coyote, once again trying to catch the Road Runner, waits for the nastily speedy bird atop a 900 foot cliff. With his Acme Rocket Pac strapped to his back, Wile E. is poised to leap from the cliff, fire up his rocket pack, and finally partake of a juicy road runner roast. Seconds later, the Road Runner zips by and Wile E. leaps from the cliff. Alas, as always, the rocket malfunctions and fails to fire, sending poor Wile E. plummeting to the road below disappearing into a cloud of dust.



a) What is the position function for Wile E. Coyote?

b) Find Wile E.'s average velocity for the first 3 seconds.

c) Find Wile E.'s average velocity between $t = 2$ and $t = 3$ seconds.

d) Find Wile E.'s velocity at the instant $t = 3$ seconds.

The problem with part *d* is that we are trying to find the *instantaneous velocity*. Without the concept of a limit, we could not find the answer to part *d*. Using a limit to solve this problem involves studying what happens to the velocity as we get "close" to 3 seconds.

Example: Find the average velocity between $t = 2.5$ and $t = 3$ seconds.

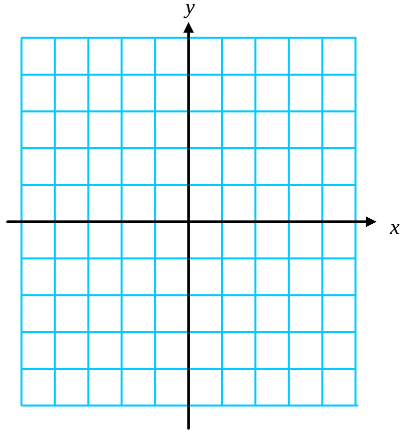
Example: Find the average velocity between $t = 2.9$ and $t = 3$ seconds.

Example: Find the average velocity between $t = 2.99$ and $t = 3$ seconds.

Example: Find the average velocity between $t = 2.999$ and $t = 3$ seconds.

So, even though we cannot find the average velocity at exactly $t = 3$ seconds, we can discover what Wile E.'s velocity is approaching at $t = 3$ seconds.

Example: Sketch the graph of $f(x) = \dfrac{x^2 - 4}{x - 2}$; $x \neq 2$.



a) What happens at $x = 2$?

b) Complete the table of values below to determine what happens as x gets “close” to 2.

x approaches 2 from the **left** ⟶ | ⟵ x approaches 2 from the **right**

X	1.5	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25	2.5
$f(x)$											

Informal Definition of a Limit

Suppose a function f is defined on an interval around $x = c$, but possibly not at the point $x = c$ itself. Suppose that as x becomes sufficiently close to c, $f(x)$ becomes as close to a single number L as we please.
We then say that the **limit of $f(x)$ as x approaches c is L**, and we write

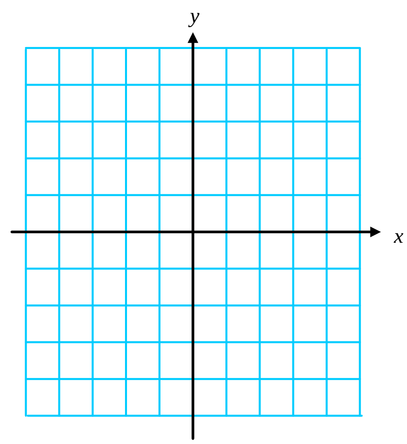
$$\lim_{x \to c} f(x) = L.$$

In other words, a limit of a function as x approaches c is defined to be the real number (y-value) that the function approaches when the x-value gets closer and closer to c.

c) Apply this definition to the function from above to find the $\lim_{x \to 2} f(x)$.

Example: Use the graph to find $\lim_{x \to 2} g(x)$, where g is defined as

$$g(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$$



So, limits exist when the y-value gets close to a specific point (even if that point isn’t actually part of the graph).

<u>When Limits Do Not Exist</u>

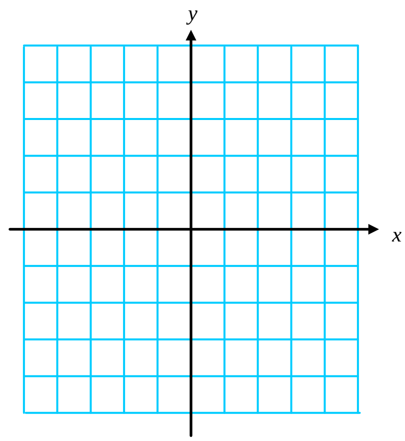
If there does not exist a <u>number</u> L satisfying the condition in the definition, then we say the $\lim_{x \to c} f(x)$ does not exist.

Limits typically fail for three reasons:

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

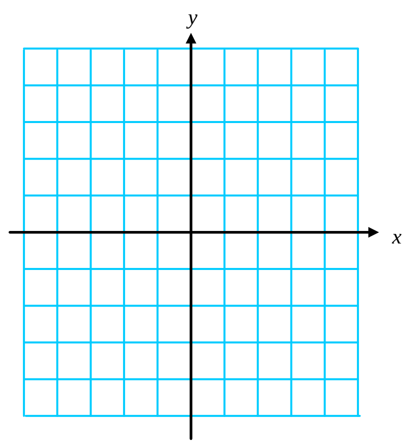
Example: Investigate (use a graph and/or table) the existence of the following limits.

(a) $\lim_{x \to 0} \frac{|x|}{x}$



X	–0.5	–0.25	–0.1	–.01	–.001	0	.001	.01	.1	.25	.5
$f(x)$											

(b) $\lim_{x \to 1} \frac{1}{(x-1)^2}$



x	0	.5	.9	.99	.999	1	1.001	1.01	1.1	1.5	2
$f(x)$											

(c) $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ … Try graphing this on your calculator.

First convince yourself that as you move to the right in the chart below x is actually getting closer and closer to 0.

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$\frac{2}{13\pi}$	As $x \to 0$
$f(x)$								

__Properties of Limits … Finding Limits By Direct Substitution__

For many "well – behaved" functions, evaluating the limit can be found by direct substitution. That is,

$$\lim_{x\to c} f(x) = f(c)$$

Such *well – behaved* functions are **continuous at *c***. We will study continuity of a function in §2.3.
The following theorems describe limits that can be evaluated by direct substitution.

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

$\lim_{x\to c} b = b$	$\lim_{x\to c} x = c$	$\lim_{x\to c}\left[f(x) \pm g(x)\right] = L \pm K$
$\lim_{x\to c}\left[f(x)\cdot g(x)\right] = LK$	$\lim_{x\to c}\left[b\cdot f(x)\right] = bL$	$\lim_{x\to c}\left[f(x)\right]^{r/s} = L^{r/s}$ provided r and s are integers and $s \neq 0$
$\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L}{K}$; provided $K \neq 0$		

Example: Use the given information to evaluate the limits: $\lim_{x\to c} f(x) = 2$ and $\lim_{x\to c} g(x) = 3$

a) $\lim_{x\to c}\left[5g(x)\right]$

b) $\lim_{x\to c}\left[f(x) + g(x)\right]$

c) $\lim_{x\to c}\left[f(x)g(x)\right]$

d) $\lim_{x\to c}\dfrac{f(x)}{g(x)}$

When dealing with a constant value for *c*, realize that the properties in the box above basically allow us to evaluate a limit by plugging in the value of *c* everywhere there is an *x*. Be careful with your variables.

Example: Find each limit

a) $\lim_{x\to 1}\left(-x^2 + 1\right)$

b) $\lim_{x\to 3}\dfrac{\sqrt{x+1}}{x-4}$

c) $\lim_{h\to 0}\left(3h^2 + 2h\right)$

d) $\lim_{h\to 0}\left(3x^2 - 2xh + 5h\right)$

<u>Other Strategies for Finding Limits</u>

If a limit cannot be found using direct substitution, then we will use other techniques to evaluate the limit.

♬: Keep in mind that some functions do not have limits.

If direct substitution yields the <u>meaningless</u> result $\frac{0}{0}$, then you ***cannot determine*** the limit in this form.

The expression that yields this result is called an **Indeterminate Form**. … *<u>DO SOMETHING ELSE!</u>*
When you encounter this form, you must rewrite the fraction so that the new <u>denominator</u> does not have 0 as its limit. One way to do this is to *cancel like factors*, and a second way is to *rationalize the <u>numerator</u>*, and a third is to simplify the algebraic expression and evaluate the limit by direct substitution again.

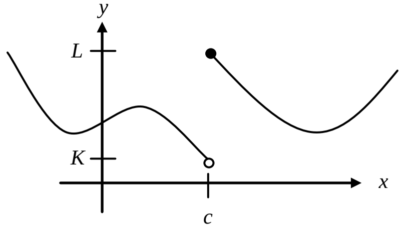
Example: Find the limit: $\lim_{x\to -1} \frac{2x^2 - x - 3}{x+1}$

Example: Find the limit (if it exists): $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$

Example: Find the limit (if it exists): $\lim_{x\to 0} \frac{\left[1/(x+4)\right]-(1/4)}{x}$

One – Sided Limits

One of the reasons a limit did not exist is because the function approached a different value from the left than it did from the right. Suppose we have the graph below.



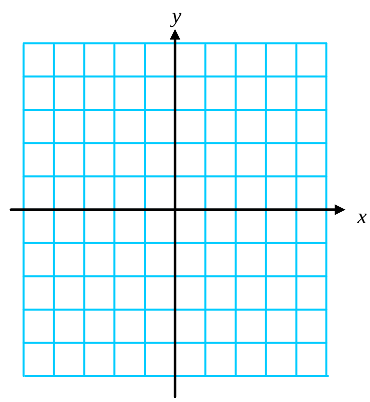
Earlier, we would have said that the limit as x approaches c does not exist because as x approaches c from the left, the function approaches K, and as x approaches c from the right, the function approaches L. However, sometimes we are interested in what the function approaches as x approaches only from the right or left of c. We can say this using the following notation:

$\lim_{x\to c^+} f(x) = L$ … "the limit of $f(x)$ as x approaches c from the right is L"
$\lim_{x\to c^-} f(x) = K$ … "the limit of $f(x)$ as x approaches c from the left is K"

Thus, we can say that the limit of a function as x approaches any number c exists <u>if and only if</u> the limit as x approaches c from the right is equal to the limit as x approaches c from the left. Using limit notation we have

$$\lim_{x\to c} f(x) \text{ exists} \Leftrightarrow \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$$

Example: Let $f(x) = \begin{cases} 5-2x & x>2 \\ 3x+1 & x\le 2 \end{cases}$



a) Graph $f(x)$

b) Find $\lim_{x\to 2^+} f(x)$.

c) Find $\lim_{x\to 2^-} f(x)$

d) What can you say about $\lim_{x\to 2} f(x)$?

Example: Investigate the $\lim_{x\to 0}\frac{\sin x}{x}$ by sketching a graph and making a table.

You must understand that while using a graph and/or a table, we may be able to determine what a limit is, we have not proved it until we algebraically confirm the limit is what we think it is. The proof of the above limit requires the use of the sandwich theorem. (See notes on pages 2-7 and 2-8) … YOU SHOULD REMEMBER THIS LIMIT!

Example: Evaluate the following limits, showing all your work where appropriate.

a) $\lim_{☺\to 0}\frac{\sin ☺}{☺}$ b) $\lim_{x\to 0}\frac{\sin 5x}{4x}$ c) $\lim_{x\to 0}\frac{\sin x}{5x^2+x}$

The Sandwich Theorem (a.k.a. The Squeeze Theorem)

The Sandwich Theorem

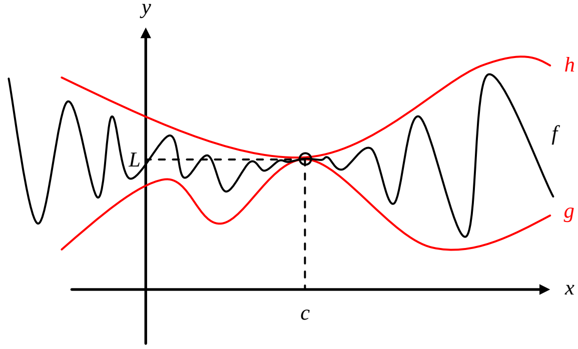
If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about *c*, and

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L,$$

then

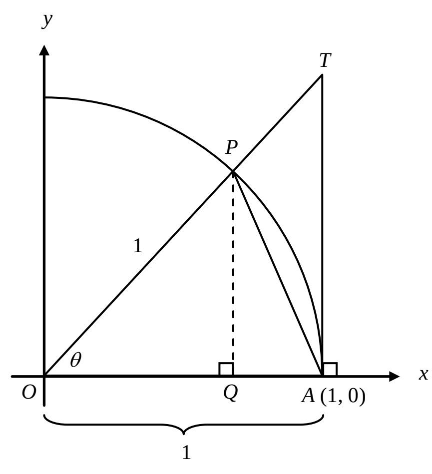
$$\lim_{x\to c} f(x) = L.$$

In other words, if we "sandwich" the function *f* between two other functions *g* and *h* that both have the same limit as *x* approaches *c*, then *f* is "forced" to have the same limit too.



Example: Prove that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$. To do this, we are going to use the figure below. Admittedly, the toughest part of using the sandwich theorem is finding two functions to use as "bread" ☺.

First, we need to find $\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}$. In order to do this we need to restrict θ so that $0<\theta<\frac{\pi}{2}$. Why are we able to do this?



a) Find the area of $\triangle OAP$.

b) Find the area of sector *OAP*.

c) Find the area of $\triangle OAT$.

d) Set up an inequality with the three areas from parts *a*, *b*, and *c*.

e) Divide all three parts by $\frac{1}{2}\sin\theta$. Why do the inequality signs stay the same?

f) Make the middle term $\frac{\sin\theta}{\theta}$. *Hint*: If your middle term doesn't look anything like this, start over! ☺

g) Use the Sandwich Theorem to show that $\lim_{x\to 0^+}\frac{\sin\theta}{\theta}=1$.

h) Show that $f(\theta)=\frac{\sin\theta}{\theta}$ is an even function.

i) Since $f(\theta)=\frac{\sin\theta}{\theta}$ is an even function, what can you conclude about $\lim_{x\to 0^-}\frac{\sin\theta}{\theta}$?

j) Explain why we can conclude that $\lim_{x\to 0}\frac{\sin\theta}{\theta}=1$.