4.4 THE FUNDAMENTAL THEOREM OF CALCULUS

The Definite Integral of a Rate Gives Total Change

Suppose a tank is being filled with water. The volume of water in a tank is $V = F(t)$ gallons. The rate of change of the volume is

$$\frac{dV}{dt} = F'(t) \text{ gallons/minute.}$$

Suppose we know the rate of change $F'(t)$, and we want to find the total amount of water that goes into the tank between $t = a$ and $t = b$. Start by dividing the interval from a to b into n equal subintervals of width

$$\Delta t = \frac{b-a}{n}$$

by inserting subdivision points

$$a = t_0, t_1, \ldots, t_{n-1}, t_n = b.$$

Example: For the first subinterval, the change in the amount of water is —

If we use the value of $F'(t)$ at the left endpoint t_0, we obtain

$$\Delta F \approx F'(t_0)\Delta t\ .$$

Example: For the second subinterval, using the value of $F'(t)$ at the left endpoint t_1, we obtain —

Repeating this procedure over all n subintervals and summing, we have

$$\begin{array}{c}\text{Total change in } F(t) \\ \text{between } t = a \text{ and } t = b\end{array} = \sum_{i=0}^{n-1} F'(t_i)\Delta t = \text{The Left-hand Sum}$$

Example: If we take the limit as $n \to \infty$ of $\sum_{i=0}^{n-1} F'(t_i)\Delta t$, what do we get?

Example: What is the TOTAL change in $F(t)$ between $t = a$ and $t = b$?

Therefore, we can view the Definite Integral of a Rate as

$$\int_a^b F'(t)\,dt = F(b) - F(a)\ .$$

This result is called the *FUNDAMENTAL THEOREM OF CALCULUS* and is usually stated as follows:

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Example: Suppose $f(t)$ represents the rate of growth of a country's gross national product in dollars per year. What are the units of $\int_0^4 f(t)\,dt$? What does this integral represent?

Example: Using the Fundamental Theorem of Calculus, evaluate the definite integral of the given functions. Use your calculator to check your answer!

(a) $\int_2^7 3\,dv$

(b) $\int_0^3 \left(3x^2 + x - 2\right) dx$

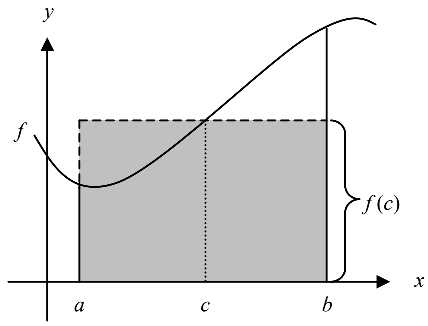
(c) $\int_1^8 \sqrt{\frac{2}{x}}\,dx$

(d) $\int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}}\,dx$

The Mean Value Theorem for Integrals

Do you remember the Mean Value Theorem we used for derivatives?

The Mean Value Theorem for Integrals basically says that if you are finding the area under a curve between $x = a$ and $x = b$, then there is *some* number c between a and b whose function value you can use to form a rectangle that has an area equal to the area under the curve.



Example: What is the area of the shaded rectangle?

Theorem 4.10 Mean Value Theorem for Integrals

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

The value of $f(c)$ in the Mean Value Theorem for Integrals is called the **Average Value** of f on the interval $[a, b]$.

Suppose $f(t)$ represents the temperature at time t, measured in hours since midnight, and suppose we want to find the average temperature over a 24 – hour period. One way to start is to measure the temperature at n equally spaced times $t_1, t_2, \ldots, t_n$ and then average those temperatures.

Example: Using this method, write an expression for the Average temperature.

The larger the number n of measurements, the more accurately this will reflect the average temperature. This expression can be written as a Riemann sum by first noting that the interval between measurements will be $\Delta t = 24/n$, so $n = 24/\Delta t$. Then

Average temperature =

As $n \to \infty$, this Riemann sum becomes an integral, thus

Average temperature =

This process can be generalized to any function *f* over an interval [*a*, *b*].

Definition of the Average Value of a Function on an Interval

If *f* is integrable on the closed interval [*a*, *b*], then the **average value** of *f* on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Example: Find the value of *c* guaranteed by the Mean Value Theorem for Integrals for the function $f(x)=\frac{9}{x^3}$ over the interval [1, 3].

The Second Fundamental Theorem of Calculus

As if one fundamental theorem was not enough. Basically, the Second Fundamental Theorem of Calculus states a relationship between the derivative and the integral. However, in order to not get mixed up with multiple uses of the letter *x*, we will integrate with respect to *t*.

Example: What do you notice about the two integrals below?

$$\int_a^b f(x)\,dx \qquad\qquad F(x)=\int_a^x f(t)\,dt$$

Theorem 4.11 The Second Fundamental Theorem of Calculus

If *f* is continuous on an open interval *I* containing *a*, then, for every *x* in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]=f(x)$$

Example: Integrate to find *F* as a function of *x*, then demonstrate the Second Fundamental Theorem of Calculus by differentiation the result.

$$F(x)=\int_4^x \sqrt{t}\,dt$$