2.3 THE PRODUCT AND QUOTIENT RULES AND HIGHER – ORDER DERIVATIVES

The Product Rule

In section 2.2 we learned that the derivative of the sum of two functions is the sum of the derivatives of the two functions. This does not work for the product and quotient of two functions. To illustrate this, we look at the following example.

Example: Find $\frac{d}{dx}\left[x\cdot x\right]$

Theorem 2.7 The Product Rule

The derivative of the product of two differentiable functions *f* and *g* is itself differentiable. Moreover, the derivative of *fg* is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}\left[f\left(x\right)g\left(x\right)\right]=f\left(x\right)g'\left(x\right)+g\left(x\right)f'\left(x\right)$$

The product rule is sometimes written with functions *u* and *v* as

$$\left(uv\right)'=uv'+vu'$$

For polynomial functions it is not always necessary to use the product rule, however, with trigonometric, exponential, logarithmic, and other functions, it is a necessary tool.

Example: Find the derivative of the following without using the product rule first, then using the product rule.

(a) $f\left(x\right)=\left(3x^3+4x^2\right)\left(2x^4-5x\right)$

Example: Find the derivative of $f\left(x\right)=x^2\cos x-5\sin x$

Theorem 2.8 The Quotient Rule

The derivative of the quotient f/g of two differentiable functions *f* and *g* is itself differentiable at all values of *x* for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}, \quad g(x) \neq 0$$

With thanks to Snow White and the Seven Dwarfs, if we replace *f*(*x*) with *hi* and *g*(*x*) with *ho* (*hi* for high up there in the numerator and *ho* for low down there in the denominator), and letting *D* stand for "the derivative of", the formula becomes

$$D\left(\frac{hi}{ho}\right) = \frac{ho\,D(hi) - hi\;D(ho)}{(ho)^2}$$

In words, that is "ho dee hi minus hi dee ho over ho ho. Now, if Sleepy and Sneezy can remember that, it shouldn't be any problem for you.

Example: Find $\frac{d}{dx}\left(\frac{x}{x^2+1}\right)$

Example: Find $\frac{d}{dx}\left[\frac{5x^2}{x^3+1}\right]$

Example: Find $\frac{d}{dx}\left[\tan x\right]$

Derivatives of Trigonometric Functions

We already knew the derivatives of sine and cosine, and in the last example we found the derivative of tan x. The rest of the trigonometric derivatives can be found in much the same way.
These derivatives are definitely worth memorizing!

Theorem 2.9 Derivatives of Trigonometric Functions

$\frac{d}{dx}[\sin x] = \cos x$ $\qquad$ $\frac{d}{dx}[\cos x] = -\sin x$

$\frac{d}{dx}[\tan x] = \sec^2 x$ $\qquad$ $\frac{d}{dx}[\cot x] = -\csc^2 x$

$\frac{d}{dx}[\sec x] = \sec x \tan x$ $\qquad$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$

Example: Differentiate $f(x) = \frac{\sec x}{1 + \tan x}$

Example: Find the equation of the tangent line to the graph of $f(x) = \frac{\tan x}{x}$ at $x = 2$.

Higher – Order Derivatives

Just as you can obtain a velocity function by differentiating a position function, you can obtain an acceleration function by differentiating a velocity function. Another way of looking at this is that you can obtain an acceleration function by differentiating a position function *twice*.

$s(t)$	Position function
$v(t) = s'(t)$	Velocity function
$a(t) = v'(t) = s''(t)$	Acceleration function

The function given by $a(t)$ is the second derivative of $s(t)$ and is denoted by $s''(t)$.

The second derivative is an example of a higher – order derivative. We can continue to take derivatives using the following notation:

First derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x\ [y]$
Second derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$	$D_x^2[y]$
Third derivative	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$	$D_x^3[y]$
Fourth derivative	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$	$D_x^4[y]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n^{th} derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$	$D_x^n[y]$

Example: Once again trying to blow up earth because it interferes with his view of Venus, Marvin the Martian lands on the moon. Bugs Bunny, as always, interferes with his plan. Chasing Bugs, Marvin fires a warning shot straight up into the air with his Acme Disintegration Pistol. If the height (in feet) after t seconds of the shot is given by

$$s(t) = -2.66t^2 + 135t + 3,$$

what is the acceleration due to gravity on the moon?